Introduction to Equivalences of Categories

Forschungsseminar: Aspects of Noncommutative Geometry

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Categories

Definition

The data of a category $\mathcal{C}$ is that of

- A collection of *objects*, denoted $\mathcal{C}_0$ or $\mathtt{ob}\,\mathcal{C}$.
- A collection of *arrows* or *morphisms*, denoted $\mathcal{C}_1$ or $\mathtt{Arr}\,\mathcal{C}$. To each arrow of $\mathcal{C}$ are associated two objects of $\mathcal{C}$, a *source* and a *target*.
- A *composition* operation $\circ$, determining for each pair of arrows $X \xrightarrow{f} Y$ and $Y \xrightarrow{g} Z$ a *composite* $g \circ f \colon X \to Z$. This is required to satisfy the following properties.
  - *Associativity*: For all triples of arrows $A \xrightarrow{f} B, B \xrightarrow{g} C, C \xrightarrow{h} D$,
  $$h \circ (g \circ f) = (h \circ g) \circ f$$
  - *Unitality*: For every object X, there is an *identity* arrow 1_X or id_X such that for arbitrary $W \xrightarrow{f} X$ and $X \xrightarrow{g} Y$ we have the equations
  $$1_X \circ f = f \qquad\qquad g \circ 1_X = g$$

Some examples of categories

Examples

Some examples of categories in the spirit that we have motivated them are

- There is the category $\mathtt{Set}$ of sets (the objects) and functions (the arrows) between them.
- There is the category $\mathtt{Top}$ of topological spaces and continuous maps between them.
- There is a category $\mathtt{Meas}$ of measurable spaces and measurable functions.
- There is the category $\mathtt{Grp}$ of groups and group homomorphisms.
- There is the category $\mathtt{Vect}_k$ of vector spaces over a field k and linear maps.
- There is a category $\mathtt{Ring}$ of (unital) rings and (unital) ring homomorphisms between them.

We will see more such examples in the following talks.

Subcategories

Definition

- A *subcategory* of a category $\mathcal{C}$ is a category defined by a subcollection of objects of $\mathcal{C}$, and a subcollection of arrows of $\mathcal{C}$ between these objects containing the respective identities, such that the composition operation of $\mathcal{C}$ restricts to one of this new category.
- A subcategory of $\mathcal{C}$ is *full* when its arrows are all arrows of $\mathcal{C}$ between its objects.

Examples

Example

There are full subcategories of interest of all the examples above. For instance,

- There is the subcategory $\texttt{FinSet}$ of $\texttt{Set}$ given by finite sets and functions between them.
- There is the subcategory $\texttt{CHaus}$ of $\texttt{Top}$ given by the compact Hausdorff spaces.
- There is the subcategory of $\texttt{Vect}_k^{\texttt{fg}}$ of $\texttt{Vect}_k$ of finitely generated vector spaces over a field k and linear maps.

Edge case: Monoids

Example

- Recall that a monoid M is a set (abusively denoted M as well) with an associative binary operation $M \times M \to M$ such that there is a unit element e for this operation.
- This can be seen as a category with one object, where the arrows are the elements of M (with source and target the unique object). The composition operation is just the multiplication operation of the monoid.

Edge case: Preorders

Example

- Recall that a *preorder* is a set with a binary operation $\leq$ that is reflexive and transitive.
- A preorder can equivalently be represented as a category whose objects are the elements of the preorder, and a unique morphism $X \to Y$ when $X \leq Y$.
- An elegant example of price providing such a preorder relation can be found in [Per19, Example 1.1.4], and similar examples can be constructed when one deals with objects that are inputs to a kind of "height function" (for the same reason).

Finite totally ordered sets

Notation

- As a particularly useful class of preorder categories, we will denote for $n \in \mathbb{N}$ the finite totally ordered set

$$[n] := \{0 < 1 < \ldots < n\}$$

- They have the particularly evocative depiction

$$0 \to 1 \to \cdots \to n$$

Isomorphisms

Definition

We call an arrow $f\colon X \to Y$ of a category $\mathcal{C}$ an *isomorphism* if it is *invertible*, i.e. if there is an arrow $g\colon Y \to X$ such that both $f \circ g$ and $g \circ f$ are the respective identities.

Example

- The invertible arrows in `Set` are the bijections.
- In many "algebraic" categories as well, invertibility and bijectivity imply each other (group/ring homomorphisms, linear maps, etc).
- However in the case of `Top` or `Meas`, this is not the case, as one would for instance require the inverses to be continuous or measurable as well.

Example

- Thinking of a monoid as a category, we see that the invertible elements of the monoid correspond to the invertible arrows.
- Thus for instance, a monoid is a group if and only if every arrow of the corresponding category is invertible.

Example

- In a preorder, two objects are isomorphic if and only if there are morphisms in both ways between them.
- Consequently two objects X, Y are isomorphic if and only if $X \leq Y$ and $Y \leq X$.

Functors

Definition

A *functor* $\mathcal{F}\colon \mathcal{C} \to \mathcal{D}$ from a category $\mathcal{C}$ to $\mathcal{D}$ is the data of

- An assignment on objects $\operatorname{ob}\mathcal{C} \to \operatorname{ob}\mathcal{D}$
- For each pair of objects X, Y an assignment on morphisms

$$\mathcal{F} := \mathcal{F}_{X,Y}\colon \mathcal{C}(X, Y) \to \mathcal{D}(\mathcal{F}X, \mathcal{F}Y)$$

such that

- $\mathcal{F}$ takes identities to identities.
- For composable $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{C}$,

$$\mathcal{F}(g \circ f) = \mathcal{F}(g) \circ \mathcal{F}(f)$$

Examples of functors

Example

- Functors between monoids and preorder categories are monoid homomorphisms and order preserving maps respectively.
- For all of our prototypical examples, there are *forgetful functors* to `Set`, which extract the underlying set and underlying function from the objects and morphisms respectively.
- The formation of the free group/vector space/polynomial ring on a set define functors out of `Set`, as given a function from one set to another, one can associate to it the unique arrow between free objects that acts as the given function on generators.

Example

For a category $\mathcal{C}$, the data of a functor $[0] \to \mathcal{C}$ corresponds to the datum of an object of $\mathcal{C}$. Similarly, a functor $[1] \to \mathcal{C}$ amounts to picking out an arrow of $\mathcal{C}$.

Example

- Thinking of a group G as a one object category, a functor $G \to \mathtt{Set}$ is determined by a set X and a group homomorphism $G \to \mathtt{Aut}_{\mathtt{Set}}(X)$.
- In other words, a functor from a group to $\mathtt{Set}$ is the data of an action of the group on a set.
- One obtains analogues of this correspondence on replacing $\mathtt{Set}$ with other categories, for instance functors to $\mathtt{Top}$ produce continuous group actions, and functors to $\mathtt{Vect}_k$ produce representations.

Definition

- Consider functors $\mathcal{F}, \mathcal{G} \colon \mathcal{C} \to \mathcal{D}$. A *natural transformation* $\mathcal{F} \to \mathcal{G}$ is the data of a functor
$$\mathcal{H} \colon [1] \times \mathcal{C} \to \mathcal{D}$$
such that the restriction of $\mathcal{H}$ to $\{0\} \times \mathcal{C}$ is $\mathcal{F}$ and its restriction to $\{1\} \times \mathcal{C}$ is $\mathcal{G}$.
- Unpacking this definition, a natural transformation $\mathcal{H} \colon \mathcal{F} \to \mathcal{G}$ is the data of an arrow $\mathcal{H}_c \colon \mathcal{F}(c) \to \mathcal{G}(c)$ of $\mathcal{D}$ for each object c of $\mathcal{C}$ such that, for each arrow $u \colon x \to y$ of $\mathcal{C}$, the following square in $\mathcal{D}$ commutes
$$\begin{array}{ccc} \mathcal{F}(x) & \xrightarrow{\mathcal{F}(u)} & \mathcal{F}(y) \\ {\scriptstyle \mathcal{H}_x}\downarrow & & \downarrow{\scriptstyle \mathcal{H}_y} \\ \mathcal{G}(x) & \xrightarrow[\mathcal{G}(u)]{} & \mathcal{G}(y) \end{array}$$
- The arrow $\mathcal{H}_c$ is called the *component* of $\mathcal{H}$ at c.

Examples of natural transformations

Example

- Thinking of a group as a one object category, and functors $G \to \texttt{Set}$ as G-sets, a natural transformation is a G-equivariant maps.
- Similarly, natural transformations between similar functors, say to topological spaces, or vector spaces are the corresponding notions of G-equivariant maps.
- On the other hand, given order preserving preorder maps $\mathcal{F}, \mathcal{G} \colon \mathcal{C} \to \mathcal{D}$, a natural transformation (and there can be at most one from a given preorder map to another) $\mathcal{F} \to \mathcal{G}$ expresses the fact that $\mathcal{F} \leq \mathcal{G}$ (objectwise).

Construction

Consider functors $\mathcal{F}, \mathcal{G}, \mathcal{H} \colon \mathcal{C} \to \mathcal{D}$, and natural transformations $\alpha \colon \mathcal{F} \to \mathcal{G}$ and $\beta \colon \mathcal{G} \to \mathcal{H}$. The (vertical) composition $\beta \circ \alpha \colon \mathcal{F} \to \mathcal{H}$ is the natural transformation whose component at an object c is

$$(\beta \circ \alpha)_c := \beta_c \circ \alpha_c$$

Natural Equivalences

Definition

Given functors $\mathcal{F}, \mathcal{G} \colon \mathcal{C} \to \mathcal{D}$, a *natural equivalence* is a natural transformation $\alpha \colon \mathcal{F} \to \mathcal{G}$ that has an inverse under vertical composition.

Observation

Given functors $\mathcal{F}, \mathcal{G} \colon \mathcal{C} \to \mathcal{D}$ and a natural transformation $\alpha \colon \mathcal{F} \to \mathcal{G}$, α is a natural equivalence if and only if each component $\alpha_c \colon \mathcal{F}(c) \to \mathcal{G}(c)$ is invertible (in $\mathcal{D}$).

Equivalences of Categories

Definition

- An *equivalence* of categories is a functor $\mathcal{F}\colon \mathcal{C} \to \mathcal{D}$ such that there exists a functor $\mathcal{G}\colon \mathcal{D} \to \mathcal{C}$, as well as natural equivalences $1_{\mathcal{C}} \cong \mathcal{G} \circ \mathcal{F}$, and $\mathcal{F} \circ \mathcal{G} \cong 1_{\mathcal{D}}$.
- In such a scenario one calls $\mathcal{G}$ a *pseudo-inverse* to $\mathcal{F}$.
- We will denote equivalences of categories as $\mathcal{F}\colon \mathcal{C} \xrightarrow{\sim} \mathcal{D}$, and in such a scenario write $\mathcal{C} \simeq \mathcal{D}$.

Proposition

One can in fact show that pseudo-inverses of an equivalence are themselves unique up to natural equivalence.

Fully faithful and essentially surjective functors

Definition

Consider categories $\mathcal{C}, \mathcal{D}$, and a functor $\mathcal{F} \colon \mathcal{C} \to \mathcal{D}$.

- We say that $\mathcal{F}$ is *faithful* if for objects X, Y of $\mathcal{C}$, the induced
$$\mathcal{C}(X, Y) \to \mathcal{D}(\mathcal{F}(X), \mathcal{F}(Y))$$
is injective.
- We say that $\mathcal{F}$ is *full* if for objects X, Y of $\mathcal{C}$, the induced
$$\mathcal{C}(X, Y) \to \mathcal{D}(\mathcal{F}(X), \mathcal{F}(Y))$$
is surjective.
- We say that $\mathcal{F}$ is *fully-faithful* if it is both full and faithful.
- We say that $\mathcal{F}$ is *essentially surjective* if every object Y of $\mathcal{D}$ is isomorphic to $\mathcal{F}(X)$ for an object X of $\mathcal{C}$.

Characterizing equivalences

Proposition

Consider categories $\mathcal{C}, \mathcal{D}$, and a functor $\mathcal{F} : \mathcal{C} \to \mathcal{D}$. Then, $\mathcal{F}$ is an equivalence if and only if it is fully-faithful and essentially surjective.

An application of the preceding criterion

Example

- For a field k, let $\mathtt{Mat}$ be the category with objects the k-vector spaces k^n for each $n \geq 0$. The arrows $k^m \to k^n$ are given by $n \times m$ matrices, and composition by matrix multiplication.
- Now, k^n has a canonical basis, and thinking of such a matrix as encoding the action of a linear map on this basis we get a functor $\mathtt{Mat} \to \mathtt{Vect}_k^{\mathtt{fg}}$.
- From classic linear algebra one knows that the information of how a linear map is characterised by its action on a basis is precisely given by a matrix as such. Thus one fully-faithfulness.
- Additionally, essential surjectivity is reflected as the fact that every vector space has a basis (which of course, we have assumed to be finite). Thus in particular this functor is an equivalence.
- Somewhat interesting is the fact that we do not have a canonical explicit pseudo-inverse to $\mathtt{Mat} \to \mathtt{Vect}_k^{\mathtt{fg}}$. This can be seen as reflecting the fact that general vector spaces do not have a canonical choice of basis.

Skeletal categories

Definition

- A category $\mathcal{C}$ is *skeletal* when no two distinct objects are isomorphic.
- A skeleton of a category $\mathcal{C}$ is a skeletal subcategory $\mathcal{A}$ such that the inclusion $\iota \colon \mathcal{A} \rightarrowtail \mathcal{C}$ is an equivalence.

Observation

In light of the characterization of equivalences, a skeleton of a category is necessarily full.

Examples of skeletal categories

Example

- Monoids as one object categories are skeletal for formal reasons.
- More interestingly, a preorder is skeletal precisely when $x \leq y$ and $y \leq x$ implies that $x = y$.
- In other words, the skeletal preorders are precisely the *posets* (partially ordered sets).
- Finally, the example of the equivalence between matrices and linear maps can be interpreted as an identification of $\mathtt{Mat}$ as a skeleton of $\mathtt{Vect}_k^{\mathtt{fg}}$.

Existence of skeleta

Proposition

Every category $\mathcal{C}$ has a skeleton, given by the full subcategory $\mathcal{A}$ spanned by a choice of representatives of each isomorphism class.

Observation

- *An equivalence of skeletal categories is an isomorphism.*
- *Further, equivalent categories have equivalent, and thus isomorphic, skeleta.*
- *Conversely, if two categories have isomorphic skeleta, they are equivalent.*

Corollary

A monotone map of posets is invertible if and only if it is order-reflecting (i.e. order-detecting) and surjective.

Further reading

- As mentioned on the course webpage, Paolo Perrone's *Notes on category theory with examples from basic mathematics* [Per19] is a fantastic introduction to category theory, and this talk has (by design or otherwise) followed it quite closely.
- The author's personal introduction to category theory was largely from Emily Riehl's *Category Theory in Context* [Rie16], which also has several examples from a more traditional mathematical perspective.
- An interesting and very self-contained introduction from a more logical/conceptual perspective can also be found in Lawvere and Schanuel's *Conceptual Mathematics* [LS09].

References

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