SEMINAR: ASPECTS OF NONCOMMUTATIVE GEOMETRY
INTRODUCTION TO EQUIVALENCES OF CATEGORIES

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Introduction

Categories are (amongst other things) a versatile tool for organising the data of how (mathematical) objects of a certain type interact with each other. These objects can in fact often be characterized in a functional way based on these interaction with other objects. Category theory provides a way to make this precise, as well as allowing us to reason about these objects in a diagrammatic way.

Prototypical examples of this kind of application in noncommutative geometry are correspondences between objects that are intuitively "topological/geometric" and those that appear more "algebraic" in nature. The notion of "interaction" between the "topological/geometric" objects will correspond to the analogues of "continuous/smooth" maps, and those between the "algebraic" objects will correspond to "homomorphisms". The purpose of this talk will be to make precise what we mean by this sort of correspondence, and what is meant by the slogan "*Algebra is dual to geometry.*".

Contents

1. Categories

As mentioned in the introduction, the fundamental data of a category comprises of objects and interactions between them. The crucial additional data is that of an *associative composition operation*.

Definition 1.1. The data of a category $\mathcal{C}$ is that of

- A collection of *objects*, denoted $\mathcal{C}_0$ or $\mathsf{ob}\,\mathcal{C}$.
- A collection of *arrows* or *morphisms*, denoted $\mathcal{C}_1$ or $\mathsf{Arr}\,\mathcal{C}$. To each arrow of $\mathcal{C}$ are associated two objects of $\mathcal{C}$, a *source* and a *target*.

Notation 1.2. We will use the notations $f\colon X \to Y$ and $X \xrightarrow{f} Y$ to denote an arrow f with source X and target Y. One often calls this an arrow *from* X *to* Y. We will also use $\mathcal{C}(X, Y)$ to denote the collection of arrows from X to Y.[1]

- A *composition* operation $\circ$, determining for each pair of arrows $X \xrightarrow{f} Y$ and $Y \xrightarrow{g} Z$ a *composite* $g \circ f\colon X \to Z$. This is required to satisfy the following properties.
  - *Associativity*: For all triples of arrows $A \xrightarrow{f} B, B \xrightarrow{g} C, C \xrightarrow{h} D$,
  $$h \circ (g \circ f) = (h \circ g) \circ f$$
  - *Unitality*: For every object X, there is an *identity* arrow 1_X or id_X such that for arbitrary $W \xrightarrow{f} X$ and $X \xrightarrow{g} Y$ we have the equations
  $$1_X \circ f = f \qquad\qquad g \circ 1_X = g$$

Remark 1.3. Some authors use $f; g : X \to Z$ to denote the composite of $X \xrightarrow{f} Y$ and $Y \xrightarrow{g} Z$. This has the advantage of more literally denoting the idea of "first f, then g" (when written left to right, at least).

Warning 1.4. In principle there are some size issues to be addressed. We have used "collection" as a placeholder word. We expect them to behave like sets, but in practice we will be interested in collections such as that of all sets, all groups, all measurable spaces, etc. Thus much like we cannot naively speak of the set of all sets, some care must be exercised in speaking of the categories of such things.

There are ways around this problem, but we will not cover them in this talk as our use of categories will not directly engage with the technicalities of these workarounds. It will be important to us that this problem can be circumvented, but not precisely how.

For the curious, a survey of some workarounds can be found in [Shu08].

Notation 1.5. A category is often depicted as a labelled, directed (multi) graph, except one usually omits the identity morphisms. One may also omit some composites if they are clear in context (for instance when there is known to be a unique morphism between two particular objects).

Examples 1.6. Some examples of categories in the spirit that we have motivated them are

- There is the category Set of sets (the objects) and functions (the arrows) between them.
- There is the category Top of topological spaces and continuous maps between them.
- There is a category Meas of measurable spaces and measurable functions.
- There is the category Grp of groups and group homomorphisms.
- There is the category Vect_k of vector spaces over a field k and linear maps.
- There is a category Ring of (unital) rings and (unital) ring homomorphisms between them.[2]

We will see more such examples in the following talks.

Definition 1.7. A *subcategory* of a category $\mathcal{C}$ is a category defined by a subcollection of objects of $\mathcal{C}$, and a subcollection of arrows of $\mathcal{C}$ between these objects containing the respective identities, such that the composition operation of $\mathcal{C}$ restricts to one of this new category.

A subcategory of $\mathcal{C}$ is *full* when its arrows are all arrows of $\mathcal{C}$ between its objects.

Examples 1.8. There are full subcategories of interest of all the examples above. For instance,

- There is the subcategory FinSet of Set given by finite sets and functions between them.
- There is the subcategory CHaus of Top given by the compact Hausdorff spaces.

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[1] Alternative notation commonly used is $\mathsf{Hom}_{\mathcal{C}}(X, Y)$

[2] The corresponding category without the unitality requirement is somewhat humorously sometimes called Rng, to emphasise the lack of *i*dentity.

- There is the subcategory of $\mathsf{Vect}_k^{\mathsf{fg}}$ of Vect_k of *finitely generated* vector spaces over a field k and linear maps.

Observation 1.9. *Given a category $\mathcal{C}$, the identity of an object X is uniquely determined.*

Proof. This is similar to the proof of the uniqueness of identities in a monoid or group (or that of the zero of a vector space). Explicitly, if e and 1_X are two candidates for the identity arrow of X,

$$e = e \circ 1_X = 1_X$$

□

Remark 1.10. So far we have considered examples of categories of mathematical objects. In fact, mathematical objects can sometimes themselves be constructed as categories. We will consider two such cases to illustrate two "edge-cases" of categories, namely, where there is a unique object and where there is at most one arrow from one object to another.

Indeed, the data of a monoid corresponds to that of a category with one object, and instancing Observation 1.9 in a monoid thought of as a one object category produces the classical proof of uniqueness of identity of a monoid.

Example 1.11. Recall that a monoid M is a set (abusively denoted M as well) with an associative binary operation $\cdot\colon M \times M \to M$ such that there is a unit element e for this operation.

This can be seen as a category with one object, where the arrows are the elements of M (with source and target the unique object). The composition operation is just the multiplication operation of the monoid.

Example 1.12. Recall that a *preorder* is a set with a binary operation $\leq$ that is reflexive and transitive.

A preorder can equivalently be represented as a category whose objects are the elements of the preorder, and a unique morphism $X \to Y$ when $X \leq Y$. An elegant example of price providing such a preorder relation can be found in [Per19, Example 1.1.4], and similar examples can be constructed when one deals with objects that are inputs to a kind of "height function" (for the same reason).

Notation 1.13. As a particularly useful class of preorder categories, we will denote for $n \in \mathbb{N}$ the finite totally ordered set

$$[n] := \{0 < 1 < \ldots < n\}$$

they have the particularly evocative depiction

$$0 \to 1 \to \cdots \to n$$

1.1. **Isomorphisms.** Before considering equivalences of categories, let us ask about the right notion of equivalence *in* a category.

Definition 1.14. We call an arrow $f\colon X \to Y$ of a category $\mathcal{C}$ an *isomorphism* if it is *invertible*, i.e. if there is an arrow $g\colon Y \to X$ such that both $f \circ g$ and $g \circ f$ are the respective identities.

This is evocative of the general philosophy in mathematics of working with objects only up to their associated type of equivalence. For instance, one works with groups up to group isomorphism, topological spaces up to homeomorphism, differentiable manifolds up to diffeomorphism, and so on.[3]

Notation 1.15. We will often denote an isomorphism as $x \xrightarrow{\simeq} y$, and write $x \cong y$ to indicate that there is an isomorphism between objects x and y.

[3]For instance, one identifies the real numbers $\mathbb{R}$ irrespective of whether they are constructed by Dedekind cuts or Cauchy sequences, as one has a bijection between the two a-priori different sets that is compatible with the algebra and metric properties of the two constructions.

Examples 1.16. The invertible arrows in Set are the bijections. In many "algebraic" categories as well, invertibility and bijectivity imply each other (group/ring homomorphisms, linear maps, etc). However in the case of Top or Meas, this is not the case, as one would for instance require the inverses to be continuous or measurable as well.

Observation 1.17. *In a preorder, two objects are isomorphic if and only if there are morphisms in both ways between them. Consequently two objects X, Y are isomorphic if and only if $X \leq Y$ and $Y \leq X$.*

Observation 1.18. *Inverses of invertible arrows in a category are unique.*

Proof. Again, this is similar to the proof of uniqueness of inverses in a group. Explicitly, if g and g' are inverses of an arrow f, we have

$$g = g \circ (f \circ g') = (g \circ f) \circ g' = g'$$

□

Observation 1.19. *Identities are invertible and compositions of invertible arrows are invertible.*

Observation 1.20. *Thinking of a monoid as a category as in Example 1.11, we see that the invertible elements of the monoid correspond to the invertible arrows.*

Thus for instance, a monoid is a group if and only if every arrow of the corresponding category is invertible.

Definition 1.21. A category is called a *groupoid* if every arrow is invertible.

Observation 1.22. *The data of a group is equivalently that of a one object groupoid.*

Construction 1.23. For every category $\mathcal{C}$, there is a subcategory $\mathcal{C}^{\simeq}$ with the same objects but only the invertible arrows called the *core* of $\mathcal{C}$.

Construction 1.24. Counter to our example of a monoid as a category with a unique object, in an arbitrary category $\mathcal{C}$ there is a *monoid of endomorphisms*

$$\mathsf{End}_{\mathcal{C}}(X) := \mathcal{C}(X, X)$$

for each object X.

Similarly, for each object there is a *group of automorphisms*

$$\mathsf{Aut}_{\mathcal{C}}(X) := \mathsf{End}_{\mathcal{C}^{\simeq}}(X)$$

the group of invertible arrows $X \to X$ (under composition).

Example 1.25. For instance, for the category Vect_k of vector spaces over a field k, the endomorphism monoid of k^n recovers the monoid of $n \times n$ matrices over k, and the automorphism group is precisely the general linear group $\mathsf{GL}(n; k)$

Construction 1.26. For categories $\mathcal{C}$ and $\mathcal{D}$, one constructs the *product category* $\mathcal{C} \times \mathcal{D}$ with objects pairs (c, d) of objects of $\mathcal{C}$ and $\mathcal{D}$, and arrows pairs (f, g) of arrows of $\mathcal{C}$ and $\mathcal{D}$. Identities and composites are constructed index-wise.

2. Functors

We have introduced categories as a tool to handle the ways in which objects relate to each other. We now extend this philosophy to categories themselves, and ask about what the right notion of arrow between categories themselves might be.

Thinking about our "edge-cases" of monoids and preorders, we would like the answer to be compatible with the preexisting notions of arrows between monoids and preorders. In particular, when applied to one object categories, it should assign endomorphisms to endomorphisms in a way compatible with composition. Also, when applied to preorder categories, it should take objects to objects in a way such that if one object maps to another, their images do as well.

Definition 2.1. A *functor* $\mathcal{F} \colon \mathcal{C} \to \mathcal{D}$ from a category $\mathcal{C}$ to $\mathcal{D}$ is the data of

- An assignment on objects $\mathsf{ob}\,\mathcal{C} \to \mathsf{ob}\,\mathcal{D}$
- For each pair of objects X, Y an assignment on morphisms

$$\mathcal{F} := \mathcal{F}_{X,Y} \colon \mathcal{C}(X, Y) \to \mathcal{D}(\mathcal{F}X, \mathcal{F}Y)$$

such that

- $\mathcal{F}$ takes identities to identities.
- For composable $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{C}$,

$$\mathcal{F}(g \circ f) = \mathcal{F}(g) \circ \mathcal{F}(f)$$

Examples 2.2. As expected, functors between monoids and preorder categories are monoid homomorphisms and order preserving maps respectively.

For all of our prototypical examples in Examples 1.6, there are *forgetful functors* to Set, which extract the underlying set and underlying function from the objects and morphisms respectively.

Additionally, several standard mathematical constructions are functorial. For instance, the formation of the free group/vector space/polynomial ring on a set define functors out of Set, as given a function from one set to another, one can associate to it the unique arrow between free objects that acts as the given function on generators.

Observation 2.3. *Functors send invertible arrows to invertible arrows.*

Example 2.4. For a category $\mathcal{C}$, the data of a functor $[0] \to \mathcal{C}$ (recall Notation 1.13) corresponds to the datum of an object of $\mathcal{C}$. Similarly, a functor $[1] \to \mathcal{C}$ amounts to picking out an arrow of $\mathcal{C}$.

More generally, a functor $[n] \to \mathcal{C}$ is the data of a sequence of n composable arrows of $\mathcal{C}$.

Example 2.5. Thinking of a group G as a one object category, a functor $G \to \mathsf{Set}$ is determined by a set X and a group homomorphism $G \to \mathsf{Aut}_{\mathsf{Set}}(X)$. In other words, a functor from a group to Set is the data of an action of the group on a set.

One obtains analogues of this correspondence on replacing Set with other categories, for instance functors to Top produce continuous group actions, and functors to Vect_k produce representations.

Example 2.6. In fact, (size issues aside) functors are themselves arrows of a category Cat whose objects are (suitably sized) categories.

2.1. **(Commutative) Diagrams.** One of the advantages of depicting categories as labeled directed multigraphs is that one can speak of diagrams *within* a category. For example, there is the classic *commutative square* diagram, which depicts two possible compositions of morphisms from an object A to an object D, and communicates that the two composites are equal.

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

Intuitively, a diagram in a category $\mathcal{C}$ can be thought of as a labeled directed multigraph where the labels on vertices correspond to objects and labels on edges correspond to arrows. Such a diagram is meant to commute when, upon following any two paths from one vertex to the other, the composites of the arrows given by the labels along the two paths are the same.

Let us split this into two parts, the directed graph itself and the "labelling". For instance, the underlying digraph of a commutative square should just the poset graph of $[1] \times [1]$.

$$\begin{array}{ccc} (0,0) & \longrightarrow & (0,1) \\ \downarrow & & \downarrow \\ (1,0) & \longrightarrow & (1,1) \end{array}$$

This intuitive description can be reformulated in terms of functoriality. Instead of working with a mere assignment of labels to vertices and edges that satisfies a commutation condition, we might imagine a category J associated to the graph, where the commutativity constraint is implicitly encoded in this associated category itself!

Definition 2.7. A *(commutative) diagram of shape* J in a category $\mathcal{C}$ is a functor

$$j\colon J \to \mathcal{C}$$

The idea here is that the structure of the diagram is entirely encoded in the "shape" J, and the commutativity condition takes the form of the functor j respecting computation.

Observation 2.8. *In this sense, a commutative square in a category* $\mathcal{C}$ *is just a diagram of shape* $[1] \times [1]$ *(recall Construction 1.26). The point is that there is ultimately only one arrow* $(0,0) \to (1,1)$.

2.2. **Opposite categories and contravariant functors.** Often one deals with what may appear like an "orientation reversing functor".

A prototypical example of this (and of the kind of functor that will come up throughout the seminar) is a functor assigning the set of "functions" on an object in question. For instance, in topology or geometry one considers the construction assigning to a space or smooth manifold the set of continuous or smooth real-valued functions. Or in a more algebraic context, one considers the construction assigning to a vector space the space of all linear functionals on it.

When attempting to model these constructions as functors it is clear what the assignment on objects ought to be, but the issue that arises is that given an arrow $X \to Y$, one intuitively expects to go from a "function" on Y to one on X by pre-composition, while it is not clear how one would get a "function" on Y from one on X.

Definition 2.9. Given a category $\mathcal{C}$, the opposite category $\mathcal{C}^{\mathsf{op}}$ is the category with the same objects, but the arrows run in the opposite direction. Explicitly, $\mathsf{ob}\,\mathcal{C}^{\mathsf{op}} = \mathsf{ob}\,\mathcal{C}$, and for objects X, Y,

$$\mathcal{C}^{\mathsf{op}}(X,Y) := \mathcal{C}(Y,X)$$

Composition of morphisms acts as in $\mathcal{C}$, merely in the other direction. The identities of $\mathcal{C}^{\mathsf{op}}$ correspond to those of $\mathcal{C}$.

Observation 2.10. *A functor* $\mathcal{C} \to \mathcal{D}$ *induces a functor* $\mathcal{C}^{\mathsf{op}} \to \mathcal{D}^{\mathsf{op}}$ *acting in the same way (except with arrows reversed in both source and target). In fact, forming the opposite category defines an involution* $(-)^{\mathsf{op}}\colon \mathsf{Cat} \to \mathsf{Cat}$.

Examples 2.11. Thinking of a monoid (or group) as a one object category, this construction recovers the standard notion of the opposite monoid (resp. group). Explicitly, this has the same elements (and identity) but the alternative multiplication operation $\cdot^{\mathsf{op}}$, such that

$$f \cdot^{\mathsf{op}} g = g \cdot f$$

For a preorder, applying this construction produces the "reverse order". Intuitively, if the preorder relation is thought of as a "lesser than (or equal to)" relation $\leq$, the reverse order construction produces the same elements preordered with the corresponding "greater than" preorder $\geq$ (so that $X \geq Y$ if and only if $Y \leq X$).

Definition 2.12. A contravariant functor $\mathcal{F}$ from a category $\mathcal{C}$ to $\mathcal{D}$ is the data of a functor

$$\mathcal{F}\colon \mathcal{C}^{\mathsf{op}} \to \mathcal{D}$$

Example 2.13. This lets us realise our constructions assigning spaces of functions as functors. For example, the construction of the ring of real valued continuous functions on a topological space can be realised as a functor

$$C(-) : \mathsf{Top}^{\mathsf{op}} \to \mathsf{CRing}$$

Example 2.14. A particularly important example of contravariant functors are presheaves. A presheaf on a category $\mathcal{C}$ is simply a functor $\mathcal{C}^{\mathsf{op}} \to \mathsf{Set}$. A general theme in modern approaches to geometry is to restate classical gluing constructions in terms of so called "sheaf criteria" (which we will encounter in our more algebro-geometric examples.)

They are so ubiquitous that many take to calling functors $\mathcal{C}^{\mathsf{op}} \to \mathcal{D}$ $\mathcal{D}$-valued presheaves on $\mathcal{C}$. The intuition captured is that presheaves assign each object c of $\mathcal{C}$ an object $\mathcal{F}(c)$ of $\mathcal{D}$, and the functor data is that of "restriction" maps $\mathcal{F}(c') \to \mathcal{F}(c)$ for each arrow $c \to c'$ in $\mathcal{C}$.

3. Natural Transformations

Intuitively, natural transformations are morphisms of diagrams, or in our conception of commutative diagrams, functors themselves. In keeping with our approach so far, this should mean that a natural transformation between functors $\mathcal{F}, \mathcal{G} \colon \mathcal{C} \to \mathcal{D}$ can be realised an arrow in some category $\mathsf{Fun}(\mathcal{C}, \mathcal{D})$ whose objects are functors $\mathcal{C} \to \mathcal{D}$.

Recall from Example 2.4 that such an arrow should then correspond to a functor

$$[1] \to \mathsf{Fun}(\mathcal{C}, \mathcal{D})$$

such that the object picked out by $0 \in [1]$ is $\mathcal{F}$ and that picked out by $1 \in [1]$ is $\mathcal{G}$.

To the homotopically minded, this may look like a (directed) path in a mapping space

$$[0,1] \to \mathsf{Map}(C, D)$$

and one knows that (topological considerations aside) this is just a homotopy of maps $C \to D$, which is equivalently given as a Map

$$H \colon [0,1] \times \mathcal{C} \to \mathcal{D}$$

We will define natural transformations in a way that makes this analogy precise.

Definition 3.1. Consider functors $\mathcal{F}, \mathcal{G} \colon \mathcal{C} \to \mathcal{D}$. A *natural transformation* $\mathcal{F} \to \mathcal{G}$ is the data of a functor

$$\mathcal{H} \colon [1] \times \mathcal{C} \to \mathcal{D}$$

such that the restriction of $\mathcal{H}$ to $\{0\} \times \mathcal{C}$ is $\mathcal{F}$ and its restriction to $\{1\} \times \mathcal{C}$ is $\mathcal{G}$.

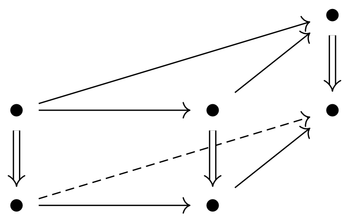
Unpacking this definition, a natural transformation $\mathcal{H} \colon \mathcal{F} \to \mathcal{G}$ is the data of an arrow $\mathcal{H}_c \colon \mathcal{F}(c) \to \mathcal{G}(c)$ of $\mathcal{D}$ for each object c of $\mathcal{C}$ such that, for each arrow $u \colon x \to y$ of $\mathcal{C}$, the following square in $\mathcal{D}$ commutes

$$\begin{array}{ccc} \mathcal{F}(x) & \xrightarrow{\mathcal{F}(u)} & \mathcal{F}(y) \\ {\scriptstyle \mathcal{H}_x}\downarrow & & \downarrow{\scriptstyle \mathcal{H}_y} \\ \mathcal{G}(x) & \xrightarrow[\mathcal{G}(u)]{} & \mathcal{G}(y) \end{array}$$

The arrow $\mathcal{H}_c$ is called the *component* of $\mathcal{H}$ at c.

The pictorial idea behind this is to think of $\mathcal{F}$ and $\mathcal{G}$ as defining $\mathcal{C}$ shaped diagrams in $\mathcal{D}$. Such a $\mathcal{H}$ then determines a sort of "bridge" between the two $\mathcal{C}$ shaped diagrams in $\mathcal{D}$, and the commuting square conditions ensure that this "big diagram" is commutative.

For instance, the picture behind a natural transformation between say, two functors out of the category [2] (which represent a pair of composable arrows in a category) looks like



Examples 3.2. Thinking of a group as a one object category, and functors $G \to \mathsf{Set}$ as G-sets, a natural transformation is a G-equivariant maps. Similarly, natural transformations between similar functors, say to topological spaces, or vector spaces are the corresponding notions of G-equivariant maps.

On the other hand, given order preserving preorder maps $\mathcal{F}, \mathcal{G} \colon \mathcal{C} \to \mathcal{D}$, a natural transformation (and there can be at most one from a given preorder map to another) $\mathcal{F} \to \mathcal{G}$ expresses the fact that $\mathcal{F} \leq \mathcal{G}$ (objectwise).

Construction 3.3. Consider functors $\mathcal{F}, \mathcal{G} \colon \mathcal{B} \to \mathcal{C}$ and a natural transformation $\alpha \colon \mathcal{F} \to \mathcal{G}$. Given functors $u \colon \mathcal{A} \to \mathcal{B}$ and $v \colon \mathcal{C} \to \mathcal{D}$, we can construct the *whiskerings* αu and $v\alpha$ of α.

- $\alpha u \colon \mathcal{F}u \to \mathcal{G}u$ is the natural transformation whose component at an object a is the arrow $\alpha_{u(a)}$ of $\mathcal{C}$.
- $v\alpha \colon v\mathcal{F} \to v\mathcal{G}$ is the natural transformation whose component at an object b is the arrow $v(\alpha_b)$ of $\mathcal{D}$.

3.1. **Functor categories.** We introduced natural transformations with the idea that they should be the arrows of a functor category $\mathsf{Fun}(\mathcal{C}, \mathcal{D})$. To make this precise it remains to specify how to compose them.

Construction 3.4. Consider functors $\mathcal{F}, \mathcal{G}, \mathcal{H} \colon \mathcal{C} \to \mathcal{D}$, and natural transformations $\alpha \colon \mathcal{F} \to \mathcal{G}$ and $\beta \colon \mathcal{G} \to \mathcal{H}$. The (vertical) composition $\beta \circ \alpha \colon \mathcal{F} \to \mathcal{H}$ is the natural transformation whose component at an object c is

$$(\beta \circ \alpha)_c := \beta_c \circ \alpha_c$$

Observation 3.5. *Vertical composition is associative, and the natural transformation whose components are all identities is itself an identity for this composition.*

Definition 3.6. For categories $\mathcal{C}$ and $\mathcal{D}$, there is a category $\mathsf{Fun}(\mathcal{C}, \mathcal{D})$ whose objects are functors $\mathcal{C} \to \mathcal{D}$, morphisms are natural transformations, and composition is given by vertical composition of natural transformations.

Examples 3.7. Thinking of a group as a one object category, the functor category $\mathsf{Fun}(G, \mathsf{Set})$ is the category of G-sets and equivariant maps. Similarly the functor category $\mathsf{Fun}(G, \mathsf{Vect}_k)$ is the category of G-representations.

On the other hand, for posets P, P', the functor category $\mathsf{Fun}(P, P')$ is itself a preorder category, the preorder of order preserving maps $P \to P'$ with the preorder relation expressing a "$\leq$"-relation pointwise.

Example 3.8. For a category $\mathcal{C}$, one denotes by $\widehat{\mathcal{C}}$ the *presheaf category* $\mathsf{Fun}(\mathcal{C}^{\mathsf{op}}, \mathsf{Set})$. The arrows in this category correspond to object indexed families of functions $\mathcal{F}(c) \to \mathcal{G}(c)$ that are compatible with "restriction".

3.2. **Natural Equivalences.** Natural equivalences are isomorphisms in functor categories.

Definition 3.9. Given functors $\mathcal{F}, \mathcal{G} \colon \mathcal{C} \to \mathcal{D}$, a *natural equivalence* is a natural transformation $\alpha \colon \mathcal{F} \to \mathcal{G}$ that has an inverse under vertical composition.

Observation 3.10. *Given functors $\mathcal{F}, \mathcal{G} \colon \mathcal{C} \to \mathcal{D}$ and a natural transformation $\alpha \colon \mathcal{F} \to \mathcal{G}$, α is a natural equivalence if and only if each component $\alpha_c \colon \mathcal{F}(c) \to \mathcal{G}(c)$ is invertible (in $\mathcal{D}$).*

Proof. If α is a natural equivalence, i.e. has an inverse $\beta\colon \mathcal{G} \to \mathcal{F}$ then each β_c is an inverse to α_c. This can be seen by considering the components at c of the sides of the equations $\alpha \circ \beta = 1_{\mathcal{D}}$ and $\beta \circ \alpha = 1_{\mathcal{C}}$.

Conversely, if each component of α is invertible, one can show that the inverses are the components of an inverse natural transformation β. The key observation here is that the inverses do indeed form a natural transformation, and then the composites of α and β both ways are identity by construction. Explicitly, we must check that for $x \xrightarrow{u} y$ in $\mathcal{C}$,

$$\begin{array}{ccc} \mathcal{G}(x) & \xrightarrow{\mathcal{F}(u)} & \mathcal{G}(y) \\ {\scriptstyle \alpha_x^{-1}}\downarrow & & \downarrow{\scriptstyle \alpha_y^{-1}} \\ \mathcal{F}(x) & \xrightarrow[\mathcal{G}(u)]{} & \mathcal{F}(y) \end{array}$$

which can be seen by composing with α_y, using the naturality of α and the cancellation property for invertible arrows. □

In a sense natural equivalences are more like homotopies than general natural transformations, as one might expect homotopies to be "reversible". Indeed, there is a way to define natural equivalences as functors, much like as with natural transformations.

Construction 3.11. Let $\mathcal{J}$ be the category with two objects $0, 1$, the identities and additionally precisely one arrow $0 \to 1$ and an arrow $1 \to 0$. These non-identity arrows must thus necessarily be inverse to each other.

It can be shown that, much as functors $[1] \to \mathcal{C}$ correspond to arrows in a category $\mathcal{C}$, functors $\mathcal{J} \to \mathcal{C}$ correspond to isomorphisms. For this reason, this category is often called the *Walking* or *free isomorphism*.

Observation 3.12. *For a categories $\mathcal{C}, \mathcal{D}$ and functors $\mathcal{F}, \mathcal{G}\colon \mathcal{C} \to \mathcal{D}$, a natural equivalence $\alpha\colon \mathcal{F} \xrightarrow{\simeq} \mathcal{G}$ corresponds to the data of a functor*

$$\alpha\colon \mathcal{J} \times \mathcal{C} \to \mathcal{D}$$

restricting to $\mathcal{F}$ on $\{0\} \times \mathcal{C}$ and $\mathcal{G}$ on $\{1\} \times \mathcal{C}$.

4. Equivalences of Categories

We have (modulo size issues) the category of categories Cat, and thus already have one way of identifying categories more general than "literal" equality, that of isomorphism.

Definition 4.1. An *isomorphism of categories* is a functor $\mathcal{F}\colon \mathcal{C} \to \mathcal{D}$ that has an inverse. That is, there exists an *inverse functor* $\mathcal{G}\colon \mathcal{D} \to \mathcal{C}$ such that $\mathcal{G} \circ \mathcal{F} = 1_{\mathcal{C}}$ and $\mathcal{F} \circ \mathcal{G} = 1_{\mathcal{D}}$.

However in practice this is often a much more restrictive condition than we want. For instance, we are often interested in statements of the sort as in "Linear maps between n-dimensional vector spaces are essentially the same as $(n \times n)$-matrices". The general phenomenon here is that we are only interested in objects up to isomorphism, and thus our notion of equivalence should allow a sort of "up to isomorphism" freedom on objects.

Definition 4.2. An *equivalence* of categories is a functor $\mathcal{F}\colon \mathcal{C} \to \mathcal{D}$ such that there exists a functor $\mathcal{G}\colon \mathcal{D} \to \mathcal{C}$, as well as natural equivalences $1_{\mathcal{C}} \cong \mathcal{G} \circ \mathcal{F}$, and $\mathcal{F} \circ \mathcal{G} \cong 1_{\mathcal{D}}$. In such a scenario one calls $\mathcal{G}$ a *pseudo-inverse* to $\mathcal{F}$. We will denote equivalences of categories as $\mathcal{F}\colon \mathcal{C} \xrightarrow{\sim} \mathcal{D}$, and in such a scenario write $\mathcal{C} \simeq \mathcal{D}$.

Observation 4.3. *Isomorphisms of categories are equivalences, and equivalences of categories are closed under composition.*

The idea here is that the pseudo-inverse $\mathcal{G}$ does not entirely invert the effect of $\mathcal{F}$, but does so up to (natural) isomorphism.

Again, this may be reminiscent of the notion of homotopy equivalence to the homotopically minded. There as well, one has the idea of an inverse, but only up to some notion of equivalence.

Proposition 4.4. *One can in fact show that pseudo-inverses of an equivalence are themselves unique up to natural equivalence.*

Examples 4.5. Monoids are equivalent as categories if and only if they are isomorphic as monoids.

Let V^* denote the dual of a vector space V, i.e. the space of linear functionals on it. There is then a canonical map $V \to V^{**}$, the evaluation map. It takes a vector v to the map sending a functional f to its evaluation $f(v)$ at v.

This can be shown to be functorial in V, and thus defines a functor $\mathsf{Vect} \to \mathsf{Vect}$. It can be shown that this restricts to an equivalence on finitely generated vector spaces.

Example 4.6. For a field k, let Mat be the category whose objects are nonnegative integers n and arrows $m \to n$ are $n \times m$ matrices, with composition being given by multiplication. The objects n here represent the standard rank n k-vector space k^n, and the matrices thus, linear maps.

This can be seen to define a functor $\mathsf{Mat} \to \mathsf{Vect}$, which factors into the category $\mathsf{Vect}_k^{\mathsf{fg}}$ of finite vector spaces. One can show that this in fact is an equivalence $\mathsf{Mat} \xrightarrow{\sim} \mathsf{Vect}_k^{\mathsf{fg}}$

Remark 4.7. It is not too much work to show this directly from the definition, but we will shortly see a criterion for a functor to be an equivalence that encapsulates a lot of the work of the individual proofs.

4.1. **Anti-equivalences.**

Definition 4.8. An *anti-equivalence* between categories $\mathcal{C}, \mathcal{D}$ is an equivalence $\mathcal{C}^{\mathsf{op}} \simeq \mathcal{D}$, or equivalently $\mathcal{C} \simeq \mathcal{D}^{\mathsf{op}}$.

An anti-equivalence is simply an equivalence with an opposite category. While from a category theoretic purpose anti-equivalences are no different from equivalences (as we can always "dualise" by taking opposite categories) many of the equivalences of categories we are interested in will be anti-equivalences.

Examples 4.9. There are the prototypical anti-equivalences between "spaces" and "algebras of functions". For instance, in the following talks we will see anti-equivalences between:

- Locally compact spaces and C^*-algebras.
- Vector bundles and finite projective modules.
- Affine varieties and finitely generated reduced commutative algebras.
- Affine schemes and commutative rings.
- Compact Riemann surfaces and algebraic function fields.

and so on.

5. Fully-Faithful and Essentially Surjective Functors

The idea behind equivalences is that they are sort of equivalences "up to isomorphism". What one might expect this to mean is that between two given objects, the ways they "interact", i.e. arrows between them correspond perfectly to arrows between their images. Additionally, while every object of the target need not itself be in the image of the equivalence, it should at least be isomorphic to something that is.

Definition 5.1. Consider categories $\mathcal{C}, \mathcal{D}$, and a functor $\mathcal{F} \colon \mathcal{C} \to \mathcal{D}$.

- We say that $\mathcal{F}$ is *faithful* if for objects X, Y of $\mathcal{C}$, the induced
$$\mathcal{C}(X, Y) \to \mathcal{D}(\mathcal{F}(X), \mathcal{F}(Y))$$
is injective.
- We say that $\mathcal{F}$ is *full* if for objects X, Y of $\mathcal{C}$, the induced
$$\mathcal{C}(X, Y) \to \mathcal{D}(\mathcal{F}(X), \mathcal{F}(Y))$$
is surjective.

- We say that $\mathcal{F}$ is *fully-faithful* if it is both full and faithful. In other words, that for objects X, Y of $\mathcal{C}$, the induced
$$\mathcal{C}(X,Y) \to \mathcal{D}(\mathcal{F}(X), \mathcal{F}(Y))$$
is bijective.
- We say that $\mathcal{F}$ is *essentially surjective* if every object Y of $\mathcal{D}$ is isomorphic to $\mathcal{F}(X)$ for an object X of $\mathcal{C}$.

Proposition 5.2. *Consider categories $\mathcal{C}, \mathcal{D}$, and a functor $\mathcal{F}\colon \mathcal{C} \to \mathcal{D}$. Then, $\mathcal{F}$ is an equivalence if and only if it is fully-faithful and essentially surjective.*

Proof. A very nice proof can be found in [Per19, Theorem 1.5.16], or [Rie16, Theorem 1.5.9].

One thing to be noted is this is an example of the kind of constructions where the foundational issues we swept under the rug start to creep out. The issue here is that to construct a pseudo-inverse to a fully-faithful and essentially surjective $\mathcal{F}\colon \mathcal{C} \to \mathcal{D}$, one would like to choose for each object d of $\mathcal{D}$, an object c of $\mathcal{C}$ and an isomorphism $\mathcal{F}(c) \cong d$. This can be somewhat questionable when the category $\mathcal{D}$ is something like the category Set of "all sets", as one would be led to construct families of "as many choices as there are sets".

However, in most workarounds to the size issues surrounding such categories, there are ways to make this kind of proposition formal, and the fundamental idea of the proof will be more or less the same. □

Example 5.3. This can, for instance be used to show Example 4.6. For a field k, k^n has a canonical basis, and from classic linear algebra one knows that the information of how a linear map is characterised by its action on a basis is precisely given by a matrix. Thus one fully-faithfulness.

Additionally, essential surjectivity is reflected as the fact that every vector space has a basis (which of course, we have assumed to be finite).

Somewhat interesting is the fact that we do not have a canonical explicit pseudo-inverse to $\mathsf{Mat} \to \mathsf{Vect}_k^{\mathsf{fg}}$. This can be seen as reflecting the fact that general vector spaces do not have a canonical choice of basis.

5.1. **Skeleta.** As remarked earlier, equivalences of categories capture the notion of a "correspondence up to isomorphism".

Definition 5.4. A category $\mathcal{C}$ is *skeletal* when no two distinct objects are isomorphic.

A skeleton of a category $\mathcal{C}$ is a skeletal subcategory $\mathcal{A}$ such that the inclusion $\iota\colon \mathcal{A} \rightarrowtail \mathcal{C}$ is an equivalence.

Observation 5.5. *In light of Proposition 5.2 characterising equivalences, a skeleton of a category is necessarily full.*

Examples 5.6. Monoids as one object categories are skeletal for formal reasons.

More interestingly, a preorder is skeletal precisely when $x \leq y$ and $y \leq x$ implies that $x = y$. In other words, the skeletal preorders are precisely the *posets* (partially ordered sets).

Finally, the Example 4.6 of the equivalence between matrices and linear maps can be interpreted as an identification of Mat as a skeleton of $\mathsf{Vect}_k^{\mathsf{fg}}$.

Proposition 5.7. *Every category $\mathcal{C}$ has a skeleton, given by the full subcategory $\mathcal{A}$ spanned by a choice of representatives of each isomorphism class.*[4]

Observation 5.8. *In light of Proposition 5.2 an equivalence of skeletal categories is an isomorphism.*

Further, equivalent categories have equivalent, and thus isomorphic, skeleta. Conversely, if two categories have isomorphic skeleta, they are equivalent.

Corollary 5.9. *A monotone map of posets is invertible if and only if it is order-reflecting (i.e. order-detecting) and surjective.*

[4]Again, this can be seen by applying Proposition 5.2.

Further Reading

As mentioned on the course webpage, [Per19] is a fantastic introduction to category theory, and this talk has (by design or otherwise) followed it quite closely. The author's personal introduction to category theory was largely from [Rie16], which also has several examples from a more traditional mathematical perspective. An interesting and very self-contained introduction from a more logical/conceptual perspective can also be found in [LS09].

References

[LS09] F. William Lawvere and Stephen H. Schanuel. *Conceptual Mathematics: A First Introduction to Categories*. Cambridge University Press, 2 edition, 2009.

[Per19] Paolo Perrone. Notes on category theory with examples from basic mathematics. URL: `https://arxiv.org/abs/1912.10642`, 2019.

[Rie16] Emily Riehl. *Category Theory in Context*. Dover, URL: `http://www.math.jhu.edu/~eriehl/context.pdf`, 2016.

[Shu08] Michael A. Shulman. Set theory for category theory. URL: `https://arxiv.org/abs/0810.1279`, 2008.