PROJECTIVE EMBEDDINGS OF COMPACT RIEMANN SURFACES

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1. A classification of projective embeddings

Recall that we established a correspondence for a complex manifold X between holomorphic maps $X \to \mathbb{P}^n(\mathbb{C})$ and isomorphism classes of line bundles on X with chosen generating sections $s_0, s_1, \dots s_n$. Precisely, we showed that there is a bijective correspondence

$$\{(\mathcal{L}, (s_0, \dots, s_n)) : \mathcal{L} \text{ a line bundle on } X, \text{ generated by } s_0, \dots, s_n\}/\sim \;\to \{\text{holomorphic } X \to \mathbb{P}^n(\mathbb{C})\}$$

Where $\sim$ is the equivalence relation of line bundle isomorphism preserving choice of sections.

Our strategy to construct projective embeddings will involve considering the maps induced by suitable line bundles such that the $s_0, \dots, s_n$ form a basis of the space of global sections.

The upshot of this will be that for any other basis $t_0, \dots, t_n$ the two maps $X \to \mathbb{P}^n(\mathbb{C})$ so defined differ by a projective automorphism given by the change of basis transformation. Thus we may talk about the space of global sections defining a projective embedding, as it does for one choice of basis if and only if it does for all choices. The following proposition gives us a characterization of when this occurs.

Remark 1.1. By a projective embedding we mean an embedding into projective space, that is an injective immersion into projective space. In particular we do not require them to be closed embeddings. However the embeddings we ultimately construct will be maps from compact spaces to separated spaces, hence necessarily closed as well.

Proposition 1.2. *Let $\mathcal{L}$ be a line bundle on a complex manifold X generated by (global) sections. Then the map $\phi\colon X \to \mathbb{P}^n(\mathbb{C})$ induced by a(ny) basis is an embedding if and only if $\mathcal{L}$ has the following properties:*

- *(The sections of) $\mathcal{L}$ separate points: For any distinct points $x \neq y$ of X, there is some section $s \in \mathcal{L}(X)$ vanishing at precisely one of the two points.*

- *(The sections of) $\mathcal{L}$ separate tangent vectors: For all points $x \in X$ the composite map of taking stalks at x and quotienting*

$$d'_x \colon \{s \in \mathcal{L}(X) : s_x \in \mathfrak{m}_x\mathcal{L}_x\} \to {}^{\mathfrak{m}_x\mathcal{L}_x}\!\big/_{\mathfrak{m}_x^2\mathcal{L}_x} \tag{1.3}$$

$$s \mapsto [s_x] \tag{1.4}$$

is a surjection.

Proof. We first show that if the sections of $\mathcal{L}$ separate points and tangent vectors, then the map defined by a basis $s_0, \ldots, s_n$ is a projective embedding. Separating points implies that the map $X \to \mathbb{P}^n(\mathbb{C})$ is injective, as if two distinct points $p \neq q$ had the same image, then all sections in a basis (and hence all sections) would have value at q nonzero scalar multiple (the scalar in question would be independent of the section) of their value at p, so no section would vanish at precisely one of the two points. Thus we need only prove that it is an immersion.

First, note that for any $x \in X$ we may identify

$$ {}^{\mathfrak{m}_x\mathcal{L}_x}\!\big/_{\mathfrak{m}_x^2\mathcal{L}_x} \cong {}^{\mathfrak{m}_x}\!\big/_{\mathfrak{m}_x^2} \otimes_{\mathcal{O}_{X,x}} \mathcal{L}_x \cong \Omega^1_X(x) \otimes_{\mathcal{O}_{X,x}} \mathcal{L}_x$$

and write the map d'_x in terms of the exterior derivative d. A section s such that $s_x \in \mathfrak{m}_x\mathcal{L}_x$ can be written in a suitable neighborhood U of x as $s = f \cdot t$ with $f \in \mathcal{O}_X(U)$ such that $f_x \in \mathfrak{m}_x$ and $t \in \mathcal{L}(U)$. Then the map d'_x sends s to $df \otimes t_x$. Further as part of our classification, we saw that there is an isomorphism $\phi^*\mathcal{O}_{\mathbb{P}^n(\mathbb{C})}(1) \cong \mathcal{L}$ under which the pullbacks of the global sections z_i of $\mathcal{O}_{\mathbb{P}^n(\mathbb{C})}(1)$ correspond to the s_i.

The point of doing this is that the property of being an immersion can equivalently be stated as the pullback $\Omega^1_{\mathbb{P}^n(\mathbb{C})}(\phi(x)) \to \Omega^1_X(x)$ being surjective for any point $x \in X$. Consider any arbitrary $x \in X$ and set $p := \phi(x)$. There is a map

$$d_1 \colon \{s \in \mathcal{O}_{\mathbb{P}^n(\mathbb{C})}(1)(\mathbb{P}^n(\mathbb{C})) : s_p \in \mathfrak{m}_p\mathcal{O}_{\mathbb{P}^n(\mathbb{C})}(1)_p\} \to \Omega^1_{\mathbb{P}^n(\mathbb{C})}(p) \otimes_{\mathcal{O}_{\mathbb{P}^n(\mathbb{C}),p}} \mathcal{O}_{\mathbb{P}^n(\mathbb{C})}(1)_p$$

defined the same way as d'_x, and d_1 is surjective (we have in fact identified $\mathcal{O}_{\mathbb{P}^n(\mathbb{C})}(1)(\mathbb{P}^n(\mathbb{C}))$ with the space of linear homogenous polynomials in $n+1$ variables, so we can think of this as a "linear approximation"). There is also the pullback map $\mathcal{O}_{\mathbb{P}^n(\mathbb{C})}(1)(\mathbb{P}^n(\mathbb{C})) \to \phi^*\mathcal{O}_{\mathbb{P}^n(\mathbb{C})}(1)(X) \cong \mathcal{L}(X)$, a surjection (since the sections z_i pull back to the sections s_i, and the s_i generate $\mathcal{L}(X)$) restricting to a surjection

$$\{s \in \mathcal{L}(X) : s_x \in \mathfrak{m}_x\mathcal{L}_x\} \to \{s \in \mathcal{O}_{\mathbb{P}^n(\mathbb{C})}(1)(\mathbb{P}^n(\mathbb{C})) : s_p \in \mathfrak{m}_p\mathcal{O}_{\mathbb{P}^n(\mathbb{C})}(1)_p\}$$

The $\mathcal{O}_{\mathbb{P}^n(\mathbb{C}),p}$-linear pullback $\Omega^1_{\mathbb{P}^n(\mathbb{C})}(p) \to \Omega^1_X(x)$ induces on taking the tensor product with $\mathcal{O}_{\mathbb{P}^n(\mathbb{C})}(1)_p$ a map

$$\begin{aligned} \gamma \colon \Omega^1_{\mathbb{P}^n(\mathbb{C})}(p) \otimes_{\mathcal{O}_{\mathbb{P}^n(\mathbb{C}),p}} \mathcal{O}_{\mathbb{P}^n(\mathbb{C})}(1)_p &\to \Omega^1_X(x) \otimes_{\mathcal{O}_{\mathbb{P}^n(\mathbb{C}),p}} \mathcal{O}_{\mathbb{P}^n(\mathbb{C})}(1)_p \\ &\cong \Omega^1_X(x) \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{X,x} \otimes_{\mathcal{O}_{\mathbb{P}^n(\mathbb{C}),p}} \mathcal{O}_{\mathbb{P}^n(\mathbb{C})}(1)_p \cong \Omega^1_X(x) \otimes_{\mathcal{O}_{X,x}} \mathcal{L}_x \end{aligned}$$

fitting into a commutative square

$$\begin{array}{ccc} \{s \in \mathcal{L}(X) : s_x \in \mathfrak{m}_x\mathcal{L}_x\} & \twoheadrightarrow & \Omega^1_{\mathbb{P}^n(\mathbb{C})}(p) \otimes_{\mathcal{O}_{\mathbb{P}^n(\mathbb{C}),p}} \mathcal{O}_{\mathbb{P}^n(\mathbb{C})}(1)_p \\ \downarrow\!\!\downarrow & & \downarrow \gamma \\ \{s \in \mathcal{O}_{\mathbb{P}^n(\mathbb{C})}(1)(\mathbb{P}^n(\mathbb{C})) : s_p \in \mathfrak{m}_p\mathcal{O}_{\mathbb{P}^n(\mathbb{C})}(1)_p\} & \xrightarrow{d'_x} & \Omega^1_X(x) \otimes_{\mathcal{O}_{X,x}} \mathcal{L}_x \end{array}$$

with both the top and left morphisms surjective. Consequently the bottom map d'_x is surjective (our separating morphisms criterion) precisely when the right map

γ is. But γ can alternatively be described as the just the map $\Omega^1_{\mathbb{P}^n(\mathbb{C})}(p) \to \Omega^1_X(x)$ itself viewed under the isomorphisms $\Omega^1_{\mathbb{P}^n(\mathbb{C})}(p) \cong \Omega^1_{\mathbb{P}^n(\mathbb{C})}(p) \otimes_{\mathcal{O}_{\mathbb{P}^n(\mathbb{C}),p}} \mathcal{O}_{\mathbb{P}^n(\mathbb{C})}(1)_p$ and $\Omega^1_X(x) \cong \Omega^1_X(x) \otimes_{\mathcal{O}_{X,x}} \mathcal{L}_x$ (they are both invertible sheaves). Thus if γ is surjective, so is the pullback map and so if the sections of $\mathcal{L}$ separate tangent vectors then the map ϕ defined is an immersion, and conversely.

To complete the proof of the converse, we must show that if the map ϕ so defined is a projective embedding then the sections of $\mathcal{L}$ separate points. Consider any distinct $p \neq q$ and let their images have coordinates $\phi(p) = (z_0 : \cdots : z_n)$ and $\phi(q) = (w_0 : \cdots : w_n)$. Injectivity of ϕ then implies that the vectors $(z_0, \ldots, z_n), (w_0, \ldots, w_n) \in \mathbb{C}^{n+1}$ are linearly independent. Thus we may find a hyperplane containing one but not the other, or equivalently we may find a linear homogenous polynomial in $n+1$ variables vanishing at one point but not the other. This pulls back to a suitable section in $\mathcal{L}(X)$ that "separates" the two points. □

Definition 1.5. When a line bundle $\mathcal{L}$ on a complex manifold X generated by (global) sections satisfies either of the two equivalent conditions in the proposition, we say that $\mathcal{L}$ is very ample. We say such an $\mathcal{L}$ is ample when there is an $N \in \mathbb{N}_{>0}$ such that for all $n \geq N$, $\mathcal{L}^{\otimes n}$ is very ample.

2. A criterion for a line bundle to be very ample

Notation 2.1. For a line bundle $\mathcal{L}$ and a divisor D on a Riemann surface X, we define

$$\mathcal{L}(D) := \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{O}_X(D)$$

called the *sheaf of meromorphic sections of $\mathcal{L}$ which are multiples of $-D$.*

Remark 2.2. We know that any line bundle on a compact Riemann surface is (up to isomorphism) of the form $\mathcal{O}_X(K)$ for some divisor K, so our definition is just $\mathcal{L}(D) \cong \mathcal{O}_X(K+D)$. In particular if D is a positive divisor (which will be the case we will consider) then $\mathcal{L}(-D) \subseteq \mathcal{L}$.

Proposition 2.3. *A line bundle $\mathcal{L}$ on a compact Riemann surface X is very ample precisely when for any points p, q of X defining divisors $p := (p) = 1 \cdot p, q := (q) = 1 \cdot q$ (which we abusively also call p and q) we have the dimension formula*

$$\dim H^0(X, \mathcal{L}(-p-q)) = \dim H^0(X, \mathcal{L}) - 2$$

Proof. Assume that the dimension formula holds for any points $p, q \in X$. Then we have the containment, $\mathcal{L}(-p-q)(X) \subsetneq \mathcal{L}(-p)(X) \subsetneq \mathcal{L}(X)$ with each containment having codimension 1 (the inequality in Ex11.1(i) shows that the codimensions of both inclusions are at most one, and hence must both be one by the dimension formula). We check first that $\mathcal{L}$ is generated by global sections and then the two criteria of the previous classification, namely that the sections $\mathcal{L}$ separates points and tangent vectors.

The map $\mathcal{L}(X) \to {}^{\mathcal{L}_p}\!/\!_{\mathfrak{m}_p \mathcal{L}_p}$ sending a section s to the class of its stalk $[s_p]$ has kernel $\mathcal{L}(-p)(X)$, so is a surjection by $\mathcal{L}(-p)(X) \subsetneq \mathcal{L}(X)$ having codimension 1. Thus $\mathcal{L}$ is generated by global sections. Further, fixing arbitrary p the proper containment $\mathcal{L}(-p-q)(X) \subsetneq \mathcal{L}(-p)(X)$ for any q implies that for any $q \neq p$, there is a section $s \in \mathcal{L}(-p)(X) \setminus \mathcal{L}(-p-q)(X)$. So $s \in \mathcal{L}(X)$ is a section vanishing at p but not at q. As $p \neq q$ were arbitrary, the sections of $\mathcal{L}$ separate points. Finally, for any point $p \in X$, the existence of an $s \in \mathcal{L}(-p)(X) \setminus \mathcal{L}(-2p)(X)$ implies the surjectivity of the composite

$$\mathcal{L}(-p)(X) \to \mathfrak{m}_x\mathcal{L}_x \to {}^{\mathfrak{m}_x\mathcal{L}_x}\!\big/_{\mathfrak{m}_x^2\mathcal{L}_x}$$

as ${}^{\mathfrak{m}_x}\!\big/_{\mathfrak{m}_x^2}$ is one dimensional over $\mathbb{C}$. Thus the sections of $\mathcal{L}$ separate tangent vectors as well.

Conversely if $\mathcal{L}$ is very ample, we may simply run the arguments in reverse to get the dimension formula (so in particular we need only separating points to get

$$\dim H^0(X, \mathcal{L}(-p-q)) = \dim H^0(X, \mathcal{L}) - 2$$

when the two points are distinct, and only separating tangent vectors when they are the same) □

3. Compact Riemann surfaces are projective

Theorem 3.1. *Let X be a compact Riemann surface. Then X admits a projective embedding.*

Proof. We find a line bundle satisfying the hypothesis of the preceding criterion. Our first observation is based on the Riemann-Roch theorem for a line bundle $\mathcal{L}$ over a compact Riemann surface X of genus g, which states that

$$\chi(X, \mathcal{L}) = 1 - g + c_1(\mathcal{L})$$

where c_1 is the first Chern class map $\operatorname{Pic}(X) \cong H^1(X, \mathcal{O}_X^\times) \to H^2(X, \mathbb{Z}) \cong \mathbb{Z}$ (recall that $c_1(\mathcal{O}_X(D)) = \deg D$). This is simply a restatement of the form in the lecture notes in light of the fact that every such line bundle $\mathcal{L}$ is of the form $\mathcal{O}_X(D)$ for some divisor D. Then for any points $p, q \in X$,

$$c_1(\mathcal{L}(-p-q)) = c_1(\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{O}_X(-p-q)) = c_1(\mathcal{L}) + c_1(\mathcal{O}_X(-p-q)) = c_1(\mathcal{L}) - 2$$

so in particular $\chi(X, \mathcal{L}(-p-q)) = \chi(X, \mathcal{L}) - 2$. Now if the first cohomology groups vanish, the Euler characteristic will agree with the dimension of the zeroth cohomology, equivalently that of the space of global sections. Thus we need only find a line bundle $\mathcal{L}$ such that $H^1(X, \mathcal{L}) = 0$ and for any points $p, q \in X$ the associated first cohomology group, $H^1(X, \mathcal{L}(-p-q)) = 0$ as well.

Our next observation is that we may restate the vanishing of these cohomology groups using Serre duality. For a line bundle $\mathcal{L}$ on a compact Riemann surface X, Serre duality asserts the nondegeneracy of a bilinear pairing ($\mathcal{O}_X(D)^\vee \cong \mathcal{O}_X(-D)$)

$$H^0(X, \mathcal{L}^\vee \otimes_{\mathcal{O}_X} \Omega_X^1) \times H^1(X, \mathcal{L}) \to \mathbb{C}$$

so the vanishing criterion is equivalent to the vanishing of $H^0(X, \mathcal{L}^\vee \otimes_{\mathcal{O}_X} \Omega_X^1)$ and $H^0(X, \mathcal{L}^\vee(p+q) \otimes_{\mathcal{O}_X} \Omega_X^1)$. We know that on a compact Riemann surface X, the line bundle $\mathcal{O}_X(D)$ has no nonzero global sections whenever the degree of D is negative. Again, we can restate that in terms of a line bundle $\mathcal{L}$ using the Chern map as $\mathcal{L}(X) = 0$ whenever $c_1(\mathcal{L}) < 0$. Then we can say that a line bundle $\mathcal{L}$ is very ample when

$$c_1(\mathcal{L}^\vee(p+q) \otimes_{\mathcal{O}_X} \Omega_X^1) = c_1(\Omega_X^1) - c_1(\mathcal{L}) + 2 < 0$$

as then $c_1(\mathcal{L}^\vee \otimes_{\mathcal{O}_X} \Omega_X^1) = c_1(\Omega_X^1) - c_1(\mathcal{L}) < 0$ as well. We have assumed that $c_1(\Omega_X^1) = 2g - 2$ so in light of this it suffices to find $\mathcal{L}$ such that $c_1(\mathcal{L}) > 2g$. But we may always do so by taking sufficiently high tensors of the bundle associated to a point divisor. In other words, the line bundle associated to any point divisor is ample. Thus any compact Riemann surface admits a very ample line bundle, and hence a projective embedding. □

Remark 3.2. We may strengthen this result and construct embeddings into $\mathbb{P}^n(\mathbb{C})$ with n given in terms of the genus g. If $g = 0$, any point divisor is already very ample in light of our Chern criterion, so as $\dim_{\mathbb{C}} \mathcal{O}_X(p)(X) = 2$ (by Riemann-Roch for instance) this defines an embedding into $\mathbb{P}^1(\mathbb{C})$ (which is necessarily biholomorphic). If $g = 1$ we may use $\mathcal{O}_X(3p)$ for any point p to get (by analogous calculation) an embedding into $\mathbb{P}^2(\mathbb{C})$ (cf. Exercise 11.1). Finally, when $g \geq 2$ we may use the "tri-canonical" bundle $(\Omega^1_X)^{\otimes 3}$ (observe now $3 \cdot 2(g-1) > 2g$) to get a canonical embedding into $\mathbb{P}^{5(g-1)-1}(\mathbb{C})$.