PROJECTIVE EMBEDDINGS AND ABELIAN FUNCTIONS

SEMINAR: *THETA FUNCTIONS, COMPLEX ABELIAN VARIETIES AND MODULI SPACES*

Contents

1. Generalities on projective embeddings of complex tori

Definition 1.1. Let U, W denote (connected) complex manifolds. A holomorphic $f\colon U \to W$ is an immersion if at each point $x \in U$ the differential/derivative $(df)(x)\colon T'_x U \to T'_{f(x)} W$ is injective.

Explicitly, if $z_1, \dots, z_m$ are coordinate functions of U in a neighborhood of x and $w_1, \dots, w_n$ are coordinate functions of W in a neighborhood of $f(x)$, then the matrix $\left(\frac{\partial f_i}{\partial z_j}(x)\right)_{i,j}$ has rank m, where f_i is defined as $f_i(z) := w_i(f(z))$ in a neighborhood of x.

Definition 1.2. Let U, W denote (connected) complex manifolds. An embedding f of U into W is an injective immersion $f\colon U \to W$ such that the image is locally closed in W.

Definition 1.3. A closed embedding is an embedding that is closed as a map of topological spaces.

Remark 1.4. An embedding of U into W is just a biholomorphic mapping onto a locally closed submanifold of W. However we have not encountered submanifolds except in the special case of open submanifolds. Thus we instead may turn this into a definition.

Definition 1.5. A submanifold U of W is a manifold such that the inclusion is an immersion. Similarly a closed submanifold is a submanifold that is a closed subspace of W.

Definition 1.6. A projective embedding of a complex manifold M is a closed embedding of M in $\mathbb{P}^{\mathrm{N}}(\mathbb{C})$ for some N.

We are interested in projective embeddings of Abelian varieties, whose underlying complex manifolds are simply complex tori. Our approach fits into a general procedure for embedding the quotient manifold of a (suitable) group action into projective space.

Notation 1.7. In this section we will denote by M a complex manifold and by G a group acting properly discontinuously and freely on M through biholomorphisms, write $X := G\backslash M$ for its quotient manifold (as in Talk 1). $\mathbb{P}^{\mathrm{N}}(\mathbb{C})$ will as usual denote complex projective space of dimension N.

We desire an embedding $X := G\backslash M \rightarrowtail \mathbb{P}^{\mathrm{N}}(\mathbb{C})$. This is determined by the composite $M \to G\backslash M \rightarrowtail \mathbb{P}^{\mathrm{N}}(\mathbb{C})$ that is G invariant. A collection of $N+1$ holomorphic functions $f_0, f_1, \ldots, f_N$ on M determines a map $(f_0, \ldots, f_N) \colon M \to \mathbb{C}^{N+1}$, and if the f_i do not simultaneously vanish at any point of M we get a holomorphic $M \to \mathbb{C}^{N+1} \setminus 0 \to \mathbb{P}^{\mathrm{N}}(\mathbb{C})$.

We ask when this descends to a map $X \to \mathbb{P}^{\mathrm{N}}(\mathbb{C})$, and this requires that for any $m \in M, g \in G$, there is a nonzero complex number $\lambda_g(m)$ such that

$$(f_0(g \cdot m), \ldots, f_N(g \cdot m)) = (f_0(m), \ldots, f_N(m))\lambda_g(m)$$

That is, each f_i satisfies $f_i(g \cdot m) = f(m)\lambda_g(m)$. This is the form of the functional equation that motivated the definition of a factor of automorphy in Talk 3, and suggests that we look for functions f_i satisfying precisely this equation when $(\lambda_g)_{g\in G}$ is a genuine factor of automorphy for $G\backslash M$.

Consider such a factor of automorphy, and let A be the subspace of $\mathscr{O}_M(M)$ comprising of functions satisfying this functional equation with respect to $(\lambda_g)_{g\in G}$. Then if A is of finite rank (equivalently, dimension) $N + 1$ over $\mathbb{C}$, choosing our $f_0, \ldots, f_N$ to be a $\mathbb{C}$-basis of A ensures that we get a holomorphic map $f \colon G\backslash M \to \mathbb{P}^{\mathrm{N}}(\mathbb{C})$.

Specializing to the case of a complex torus $X = Z/L$, our $f \colon X \to \mathbb{P}^{\mathrm{N}}(\mathbb{C})$ is further a map from a (quasi-)compact space to a Hausdorff (i.e. separated) space, and hence is also closed. Thus in order to see that it is an embedding it suffices to check that both f and df are injective (i.e. that df is an injective linear transformation at each point).

We will soon show that for a (polarized) Abelian variety there is always a suitable A, such that the map f defined as above will indeed be an embedding. We will in fact define A to be a subspace of theta functions. Observe also that an invertible linear $\mathbb{C}^{N+1} \to \mathbb{C}^{N+1}$ induces a biholomorphism $\mathbb{P}^{\mathrm{N}}(\mathbb{C}) \to \mathbb{P}^{\mathrm{N}}(\mathbb{C})$ (called a "projective transformation"). Any two embeddings induced by choices of bases as above will differ by such a projective transformation (precisely the one determined by the change of base matrix corresponding to those two bases of A). So in such a situation we say that A induces a projective embedding of X.

Remark 1.8. We saw in Exercise 6.1 of the Cohomology of Sheaves 2 problem sets that holomorphic maps $X \to \mathbb{P}^{\mathrm{N}}(\mathbb{C})$ for a complex manifold X correspond to isomorphism classes of pairs of line bundles and $N+1$ generating (global) sections. Theta functions are global sections (cf. Section 3 of Talk 3) of such line bundles on Z/L, and our construction will in fact fit into this formalism. However thanks to our explicit description of theta functions on Z we may make more elementary statements (and more importantly, calculations) in the case under consideration.

2. Projective Embeddings of Polarized Abelian Varieties

Theorem 2.1. *Let $S = S(Q, l, \psi)$ be the ring of theta functions of a polarized abelian variety $(Z/L, (kH)_{k>0})$ as defined in Talk 8 (so in particular $H := Her(Q)$*

is non-degenerate). Then $S_m = \mathscr{L}_{(mQ,ml,\psi^m)}(Z/L)$ induces a projective embedding of Z/L for every $m \geq 3$.

Remark 2.2. We follow primarily the proof in [Mumford, "Theorem of Lefschetz", Chapter 1 - Page 28 of the second edition/Page 30 of the first] with some influence from [Igusa, Section III.7]

Proof. We specialize the discussion above to the case where the factor of automorphy (λ_g) is the one determined by (mQ, ml, ψ^m), so that A is now the space $\mathscr{L}_{(mQ,ml,\psi^m)}(Z/L)$. Firstly, we have seen that the space A is finite dimensional. We must now show that if $\Theta_0, \dots, \Theta_N$ is a basis, the Θ_i define an embedding.

Our next requirement was that the Θ_i do not vanish simultaneously at any point. As they were chosen to be a basis of A, this is equivalent to there being a $\Theta \in A$ for each point $x \in Z/L$ that does not vanish at x.

Lemma 2.3. *Consider an element θ of $\mathscr{L}_{(Q,l,\psi)}(Z/L)$. For any k points $a_1, \dots, a_k$ of Z such that $\sum_{i=1}^k a_i = 0$, the function $\Theta(z) := \prod_{i=1}^k \theta(z+a_i)$ defines an element of $\mathscr{L}_{(kQ,kl,\psi^k)}(Z/L)$. Further, if $\theta \neq 0$ and $k \geq 2$ we may choose for any $a \in Z$ suitable a_i as above such that the function Θ so defined is non-vanishing at a.*

Proof. Note that we can write the factor of automorphy (which we now denote by $u_t(z)$, to fit previously established notation) corresponding to (Q, l, ψ) in the form (where $\mathbf{e}(z) := e^{2\pi i z}$ as before) $u_t(z) = \mathbf{e}(Q_l(z) + c_t) = \phi(t)\,\mathbf{e}(\frac{Q(z,t)}{2i} + \frac{Q(t,t)}{4i} + l(t))$ for $z \in Z, t \in L$, where $Q_t(z) = \frac{Q(z,t)}{2i}$ is a $\mathbb{C}$-linear form and $c_t \in \mathbb{C}$ is a constant for each $t \in L$ (see Definition 3.1 of Talk 3 and Theorem 1 of Talk 5). Thus for $t \in L$, Θ satisfies the transformation relation

$$\Theta(z+t) = \Theta(z) \cdot (\prod_{i=1}^k u_t(z+a_i))$$

and

$$\prod_{i=1}^k u_t(z+a_i) = \mathbf{e}(\sum_{i=1}^k Q_t(z+a_i) + c_t) = \mathbf{e}(k(Q_t(z) + c_t))$$

where the last inequality is due to the linearity of Q_t. This shows the first assertion.

For the second, recall that if $\theta \neq 0$ then the closed subspace $V := \theta^{-1}(0) - a$ contains no nonempty open subspaces of Z (by the "Identity theorem", Corollary 2.3 of Talk 1), or in other words its complement is a dense open subspace. Thus the open map $(Z \setminus V)^{k-1} \to Z$ defined by $(x_2, \dots, x_k) \mapsto -(x_2 + x_3 + \dots + x_k)$ has image not contained in V. Let a_1 be a point in the image not contained in V and $(a_2, \dots, a_k)$ be a preimage. Then $a_1 + a_2 + \dots + a_k = 0$ and each $a + a_i \notin \theta^{-1}(0)$, so $\Theta(a) = \prod \theta(a+a_i) \neq 0$ as desired. $\square$

As a consequence, S_m does in fact induce a holomorphic map to $\mathbb{P}^N(\mathbb{C})$ as in the schema described in the previous section (in fact we also see that it suffices to have $m \geq 2$ to define holomorphic maps in this manner).

Our next step is to show that the map f so defined is injective. Assume for the purpose of contradiction that it is not, so there exist $z, w \in Z$ such that $z - w \notin L$ (i.e. z, w define different points of Z/L) and

$$(\Theta_0(z) : \dots : \Theta_N(z)) = (\Theta_0(w) : \dots : \Theta_N(w))$$

That is, for some $\lambda \neq 0$ each $\Theta_i(z) = \lambda\Theta_i(w)$. Then in fact $\Theta(z) = \lambda\Theta(w)$ for every $\Theta \in A$ since the Θ_i are a spanning set.

In particular, we can apply the above lemma for any $a, a_2, \dots a_{m-1} \in Z$ to get the relation

$$\theta(z-a)\theta(z-a_2)\dots\theta(z+a+a_2+\dots+a_{m-1}) = \lambda\theta(w-a)\theta(w-a_2)\dots\theta(w+a+a_2+\dots+a_{m-1})$$

for any particular $\theta \in \mathscr{L}_{(Q,l,\psi)}(Z/L)$. We consider this as an equation between functions in a for fixed $a_2, \dots, a_{m-1}$. We may apply logarithmic differentiation to eliminate all factors not involving a, and write

$$\omega(z+a+a_2+\dots+a_{m-1}) - \omega(z-a) = \omega(w+a+a_2+\dots+a_{m-1}) - \omega(w-a)$$

where ω denotes the logarithmic (meromorphic) differential $d(\log\theta) = \frac{d\theta}{\theta}$. Then we see that $\omega(z+a) - \omega(w+a)$ is a translation invariant form (cf. Definition 4.4 of Talks 1-2), so we can write it as dl for $l = l(a)$ a (homogenous) linear polynomial in the coordinates of a. So we have that the logarithmic differential of $\frac{\theta(z+a)}{\theta(w+a)}$ is of the form dl. There is a nonempty open connected set where $\frac{\theta(z+a)}{\theta(w+a)} = e^{\phi(a)}$ for some holomorphic ϕ, and then the logarithmic differential is just $d\phi$. So we have $d\phi = dl$ locally, and hence $\phi = l+k_1$ for some constant k_1. Hence $\theta(z+a) = k\theta(w+a)e^{l(a)}$ for some nonzero constant k on that neighbourhood. This is an equality on a nonempty open connected set between holomorphic functions on Z, hence the equality holds on all of Z by the Identity theorem again. Thus, reparametrizing by setting $y := w+a$ we have by linearity of l the relation $\theta(y+(z-w)) = \eta e^{l(y)}\theta(y)$ for some nonzero constant η. Thus, for any θ, the function $\theta(y+(z-w))$ differs from it by a unit, and $z - w \notin L$. A consequence of Theorem 3 of Talk 5 is that the space of all theta functions θ as above such that $\theta(y)$ differs from $\theta(y+\delta)$ by a unit (where $\delta = w - z \notin L$) is contained in the union of finitely many proper subspaces of $\mathscr{L}_{(Q,l,\psi)}(Z/L)$. Thus there is at least one such θ where this cannot happen, a contradiction. Thus f is injective.

It remains only to show that df is injective. The projection $Z \to Z/L$ is a local biholomorphism so it suffices to show that the composite $Z \to \mathbb{P}^N(\mathbb{C})$ has injective derivative at each point. Assume that df is not injective at some $z \in Z$, that is some tangent vector $\sum_{i=1}^{g} \alpha_i \frac{\partial}{\partial z_i}$ at z is mapped to 0. In particular, it yields 0 when applied to the coordinate functions. $f(z) = (\Theta_0(z) : \dots : \Theta_N(z))$ with some $\Theta_i(z) \neq 0$, for simplicity assume $i = 0$, the other cases are identical up to re-indexing. Then the $\frac{\Theta_i}{\Theta_0}, i > 0$ are coordinate functions in a neighbourhood of $f(z)$, and we compute for each $i > 0$

$$\sum_{j=1}^{g} \frac{\alpha_i}{\Theta_0(z)^2}\big(\Theta_0(z)\frac{\partial\Theta_i}{\partial z_j}(z) - \Theta_i(z)\frac{\partial\Theta_0}{\partial z_j}(z)\big) = 0$$

so if we define $\alpha_0 := \frac{-1}{\Theta_0(z)}\sum_{j=1}^{g}\alpha_j\frac{\partial\Theta_0}{\partial z_j}(z)$

$$\alpha_0\phi(z) + \sum_{j=1}^{g}\alpha_j\frac{\partial\phi}{\partial z_j}(z) = 0$$

when $\phi = \Theta_i$ for arbitrary i (the case $i = 0$ is tautological). Thus it is in fact true for arbitrary $\phi \in \mathscr{L}_{(mQ,ml,\psi^m)}(Z/L)$. So we write $D(\log\phi)(z) = -\alpha_0$, where $D(\log\phi) = \frac{D\phi}{\phi}$ denotes the logarithmic (directional) derivative as before.

Again, we consider the theta functions

$$\phi(w) = \theta(w-a)\theta(w-b)\theta(w-a_3)\dots\theta(w+a+b+a_3+\cdots+a_{m-1}))$$

for $a, b, a_3, \dots a_{m-1} \in Z$. If we define $h(w) := D(\log\theta)(w) \colon Z \to \mathbb{C}$, then we get

$$h(z-a)+h(z-b)+h(z-a_3)+\cdots+h(z-a_{m-1})+h(z+a+b+a_3+\cdots+a_{m-1}) = -\alpha_0$$

for all $a, b, a_3, \dots a_{m-1} \in Z$. We may in fact set $a_i = 0$ for all $i \geq 3$ and obtain some $k \in \mathbb{C}$ an equation

$$h(z-a) + h(z-b) + h(z+a+b) = k$$

for all $a, b \in Z$. Consider this as an equation of functions in a for arbitrary fixed b, then (denoting the coordinate functions by a_i) we may differentiate with respect to a_i and observe that

$$\frac{\partial h}{\partial a_i}(z+a+b) = \frac{\partial h}{\partial a_i}(z-a)$$

for arbitrary b and hence that the partial derivatives of h are all constant. It then follows that $h(w) = \sum_{i=1}^{g} c_i w_i + s$ for some fixed $c_1, c_2, \dots c_g, s \in \mathbb{C}$ (where w_i are the coordinates of w).

Lemma 2.4. *There is an $\alpha \in Z \setminus 0$ and $c \in \mathbb{C}$ such that for all $\lambda \in \mathbb{C}, w \in Z$,*

$$\theta(w+\lambda\alpha) = e^{c\lambda^2+\lambda h(w)}\theta(w)$$

Proof. It suffices to show equality for any arbitrary fixed $w \in Z$. For any c, the logarithmic derivative of $e^{c\lambda^2+\lambda h(w)}\theta(w)$ with respect to λ is $2c\lambda + h(w)$. We set $\alpha := (\alpha_1, \alpha_2, \dots, \alpha_g) \neq 0$ and $c := \frac{1}{2}(h(\alpha) - s) = \frac{1}{2}\sum_{i=1}^{g} c_i\alpha_i$ so that $2c\lambda + h(w) = h(w+\lambda\alpha)$. Now the logarithmic derivative of $\theta(w+\lambda\alpha)$ with respect to λ is by the chain rule $\sum_{i=1}^{g} \alpha_i\big(\frac{\partial}{\partial z_i}(\log\theta)\big)(w+\lambda\alpha) = h(w+\lambda\alpha)$. Thus the logarithmic derivatives of both sides agree and so as before there is an $\eta \in \mathbb{C}^\times$ such that $\theta(w+\lambda\alpha) = \eta e^{c\lambda^2+\lambda h(w)}\theta(w)$ holds for all λ. Considering the case $\lambda = 0$ shows that $\eta = 1$ when $\theta(w) \neq 0$. When it does vanish we may take $\eta = 1$ anyway to write $\theta(w+\lambda\alpha) = e^{c\lambda^2+\lambda h(w)}\theta(w)$ for all $\lambda \in \mathbb{C}, w \in Z$. $\square$

Thus we see that for all $\lambda \in \mathbb{C}$, $\theta(w+\lambda\alpha)$ differs from $\theta(w)$ by a unit. This contradicts Theorem 3 of Talk 5 as well. Consequently the derivative df is injective at each point, completing the proof.

$\square$

Corollary 2.5. *In the scenario of the theorem, let (Q', l', ϕ') be any type of a theta function for Z/L such that $Her(Q') = mH$. Then $\mathscr{L}_{(Q',l',\psi')}(Z/L)$ defines a projective embedding of Z/L as well.*

Proof. The theorem asserts that a basis of $\mathscr{L}_{(mQ,ml,\psi^m)}(Z/L)$ induces a projective embedding of Z/L. We have seen in Theorem 3.5 of Talk 8 that there is an isomorphism $\mathscr{L}_{(mQ,ml,\psi^m)}(Z/L) \cong \mathscr{L}_{(Q',l',\psi')}(Z/L)$ that has form $\theta(z) \mapsto \eta(z)\theta(z+a)$ where η is a trivial theta function, and in particular a unit. Thus if (a basis of) one defines a projective embedding, so does (a basis of) the other. $\square$

Remark 2.6. From the above corollary onwards, we return to following [Igusa]

3. A (Polarized) Abelian Variety is a complex projective variety

Definition 3.1. A Zariski closed set of $\mathbb{K}^n$ is the set of points $(x_1, \ldots, x_n)$ defined by the simultaneous vanishing of a collection of polynomials over $\mathbb{K}$ in n symbols. We write

$$V(\{f_i\}_{i\in I}) = \{x \in \mathbb{K}^n : \forall i \in I,\ f_i(x) = 0\}$$

We also call them affine Zariski-closed sets.

Observation 3.2. *The vanishing set of a collection of polynomials is the same as the vanishing set of the ideal they generate, that is*

$$V(\{f_i\}_{i\in I}) = V(\langle\{f_i\}_{i\in I}\rangle)$$

The Hilbert basis theorem asserts that $\mathbb{K}[X_1, \ldots, X_n]$ *is noetherian, so in particular we can always define an affine Zariski-closed set by finitely many polynomials.*

Definition 3.3. Let $X \subseteq \mathbb{K}^n$. Define $I(X)$ to be the ideal of $\mathbb{K}[X_1, \ldots, X_n]$ defined by the collection of polynomials vanishing at all points of X. Observe that this ideal is always reduced.

We will assume that $\mathbb{K}$ is algebraically closed so that we may use Hilbert's Nullstellensatz.

Theorem 3.4. *Hilbert's Nullstellensatz: For an ideal* $\mathfrak{a}$ *of* $\mathbb{K}[X_1, \ldots, X_n]$ *with* $\mathbb{K}$ *algebraically closed,*

$$I(V(\mathfrak{a})) = \sqrt{\mathfrak{a}} := \{f : \exists n \in \mathbb{Z}_{>0},\ f^n \in \mathfrak{a}\}$$

Corollary 3.5. I *and* V *define an order reversing bijection between Zariski-closed sets in* $\mathbb{K}^n$ *and reduced ideals of* $\mathbb{K}[X_1, \ldots, X_n]$.

Corollary 3.6. *For any subset* $X \subseteq \mathbb{K}^n$, $V(I(X))$ *is the smallest Zariski-closed set containing it.*

Remark 3.7. The affine Zariski-closed sets are the closed sets of a topology called the Zariski topology on $\mathbb{K}^n$.

When $\mathbb{K} = \mathbb{C}$, as polynomials define continuous functions all affine Zariski-closed sets are closed in the Euclidean/Analytic topology.

Definition 3.8. An affine Zariski-closed set X is called an irreducible affine variety or just affine variety when the ideal $I(X)$ is prime.

In this case the transcendence degree over $\mathbb{K}$ of the quotient field of $\mathbb{K}[X_1, \ldots, X_n]/I(X)$ is called the dimension of X, denoted $\dim(X)$.

Observation 3.9. *An affine Zariski-closed set* X *is an (irreducible) affine variety precisely when it is topologically irreducible (that is, it is not the union of two proper closed subspaces). Every affine Zariski-closed set can be represented uniquely as a finite union of affine varieties such that none of them strictly contain each other (called irreducible components). We define* $\dim(X)$ *to be the maximum of the dimensions of the irreducible components.*

Definition 3.10. Let X be an affine variety of dimension d in $\mathbb{K}^n$. Then if $I(X) = (P_1, \ldots, P_m)$, we say that a point a of X is simple (this notion is also often called "smooth" or "non-singular") if the "Jacobian at a" $(\frac{\partial P_i}{\partial z_j}(a))_{i,j}$ has rank $n-d$. This property is in fact well defined independent of the choice of generators P_i.

Proposition 3.11. *Let X be an affine variety of dimension d in $\mathbb{K}^n$ and $I(X) = (P_1, \ldots, P_m)$. The rank of the Jacobian at a point of X is always at most $n - d$. Thus the simple points are precisely those points where this rank is "maximal".*

Observation 3.12. *Let X be an affine variety of dimension d in $\mathbb{K}^n$ and $I(X) = (P_1, \ldots, P_m)$. We know that the rank of an $m \times n$ matrix is r precisely when all $(r+1) \times (r+1)$ minors vanish but some $r \times r$ minor does not. The Jacobian can be written as a matrix of polynomial functions, so if we define an affine Zariski-closed set Y by the vanishing of the $(n - d) \times (n - d)$ minors, the set $X_s := X \setminus Y$ is precisely the set of simple points of X.*

In particular, the set of simple points is the difference of X and a Zariski-closed subspace.

Definition 3.13. A Zariski closed subset of $\mathbb{P}^{\mathrm{N}}(\mathbb{K})$ is the set of points (say with homogenous coordinates $(x_0, \ldots, x_N)$) defined by the vanishing a collection of homogenous polynomials over $\mathbb{K}$ in $N+1$ symbols $X_0, \ldots, X_N$ (the notion of vanishing is well defined as the polynomials are homogenous). We write

$$V(\{f_i\}_{i \in I}) = \{x \in \mathbb{P}^{\mathrm{N}}(\mathbb{K}) : \forall i \in I,\ f_i(x) = 0\}$$

We also call them projective Zariski-closed sets.

Observation 3.14. *Again, the vanishing set of a collection of polynomials is the same as the vanishing set of the ideal they generate, that is*

$$V(\{f_i\}_{i \in I}) = V(\langle \{f_i\}_{i \in I} \rangle)$$

and we can find a finite collection of homogenous generators for the ideal.

Remark 3.15. The projective Zariski-closed sets are also the closed sets of a topology called the Zariski topology on $\mathbb{P}^{\mathrm{N}}(\mathbb{K})$. All projective Zariski-closed sets in $\mathbb{P}^{\mathrm{N}}(\mathbb{C})$ are again closed in the Euclidean/Analytic topology.

Definition 3.16. Let $X \subseteq \mathbb{P}^{\mathrm{N}}(\mathbb{K})$. Define $I(X)$ to be the homogenous ideal of $\mathbb{K}[X_0, \ldots, X_N]$ generated by the collection of homogenous polynomials vanishing at all points of X.

Definition 3.17. Let X be a projective Zariski-closed set in $\mathbb{P}^{\mathrm{N}}(\mathbb{K})$, and let $\pi \colon \mathbb{K}^{N+1} \setminus 0 \to \mathbb{P}^{\mathrm{N}}(\mathbb{K})$ denote the projection. Then $\tilde{X} := \pi^{-1}(X) \cup 0 \subseteq \mathbb{K}^{N+1}$ is precisely the affine Zariski-closed set defined by $I(X)$. We call $\tilde{X}$ the cone over X. Further, we define $\dim(X) := \dim(\tilde{X}) - 1$.

Proposition 3.18. *Unlike the affine case, V and I do not define a correspondence between varieties and reduced homogenous ideals. However if we exclude the ideal $\mathbb{K}[X_0, \ldots, X_n]_+ = (X_0, \ldots, X_n)$ of $\mathbb{K}[X_0, \ldots, X_n]$ which along with $\mathbb{K}[X_0, \ldots, X_n]$ itself both cut out the empty set we do in fact get a bijection. In particular for $X \subseteq \mathbb{P}^{\mathrm{N}}(\mathbb{K})$, $V(I(X))$ is still the smallest projective Zariski-closed set containing X.*

Definition 3.19. A projective Zariski-closed set X is called a projective variety when the ideal $I(X)$ is prime. This is equivalent to the cone $\tilde{X}$ over X being an affine variety.

Definition 3.20. A point x of a projective variety X is simple when some $\tilde{x} \in \pi^{-1}(x)$ is a simple point of $\tilde{X}$. This is equivalent to every $\tilde{x} \in \pi^{-1}(x)$ being a simple point of $\tilde{X}$.

We state the following theorem without proof, however a discussion on the idea of the proof can be found in [Igusa, Proposition 3, Section III.4] .

Theorem 3.21. *If X is an (irreducible) affine variety of dimension d in $\mathbb{C}^n$, the subset of simple points X_s is a dense connected open subset of X that is a complex submanifold of $\mathbb{C}^n$ of dimension d.*

Corollary 3.22. *If X is a projective variety of dimension d in $\mathbb{P}^{\mathrm{N}}(\mathbb{C})$, the set X_s of simple points is a dense open connected subspace of X that is a complex submanifold of $\mathbb{P}^{\mathrm{N}}(\mathbb{C})$ of dimension d.*

Remark 3.23. The property of being a submanifold is a familiar one, as much like in the real case, the variety defined by a finite set of polynomials can be seen as a level set (preimage of a point) of a function whose components are polynomial functions. The simpleness/smoothness/non-singularity condition then asserts that the function defining the variety satisfies the hypotheses of the (complex) implicit function theorem (cf. [Griffiths-Harris, Page 19,22]). However the connectivity result is a feature of the complex case, as it is not true in general in the real case (for example consider the hyperbola $x^2 - y^2 = 1$ in $\mathbb{R}^2$).

Theorem 3.24. *Assume that a basis of $\mathscr{L}_{(Q,l,\psi)}(Z/L)$ defines a projective embedding of Z/L as above, say of the form $z \mapsto (\theta_0(z) : \cdots : \theta_N(z))$ (such that $\theta_0, \ldots, \theta_N$ is a basis of $\mathscr{L}_{(Q,l,\psi)}(Z/L)$). Then, the image X of the embedding is a projective variety of dimension $g := \operatorname{rank}_{\mathbb{C}} Z = \dim_{\mathbb{C}} Z$.*

Proof. Recall that the ring generated by the theta functions $R = \mathbb{C}[\theta_0, \ldots, \theta_N]$ is a graded integral domain. This is the image of $\mathbb{C}[T_0, \ldots, T_N] \to \mathbb{C}[\theta_0, \ldots, \theta_N]$ sending $T_i \mapsto \theta_i$, so the kernel K is a homogenous prime ideal. Further, a homogenous polynomial P is contained in K if and only if $P(\theta_0, \ldots, \theta_N)$ vanishes as a function on Z, that is P vanishes on the image X. Consequently if we define $Y := V(K)$, Y is the smallest Zariski-closed set containing X and is a projective variety.

We first show $X = Y$, so that X is a projective variety. If we set $A := Im(H)$ (where $H := Her(Q)$ is the nondegenerate Hermetian form as in the last section) we know that $\operatorname{rank}_{\mathbb{C}}(R_k) \leq \operatorname{Pfaff}(A) k^g$ for $k \in \mathbb{Z}_{>0}$.

[Igusa, Proposition 1, Section III.2] asserts that any $g + 2$ homogenous elements of a graded integral domain S of a field $\mathbb{K}$ such that each S_k is finitely generated over S_0 (the "componentwise finiteness" condition) and satisfying the asymptotic growth condition

$$\dim_{\mathbb{K}}(S_k) \prec k^g$$

are algebraically dependent over $\mathbb{K}$.

In the situation at hand, this implies that any $g + 2$ homogenous elements of R are algebraically dependent over $\mathbb{C}$. This implies as in the remark immediately after Proposition 1 that the dimension d of Y is at most g (recall that the dimension of a projective variety is 1 less than the dimension of the "cone" over it).

Let Y_s denote the simple points of Y, so that it is a dense connected open subspace of Y that is a d dimensional complex submanifold of $\mathbb{P}^{\mathrm{N}}(\mathbb{C})$. Further as it is the difference of Y and a projective Zariski-closed set in $\mathbb{P}^{\mathrm{N}}(\mathbb{C})$, and Y is the smallest projective Zariski-closed set containing X there is a point $x \in X \cap Y_s$. X is compact and hence closed in $\mathbb{P}^{\mathrm{N}}(\mathbb{C})$, hence $X \cap Y_s$ is closed in Y_s.

We show that it is also open in Y_s. For any arbitrary point $x \in X \cap Y_s$ we may find a neighbourhood V of x in $\mathbb{P}^{\mathrm{N}}(\mathbb{C})$ such that $Y \cap V = Y_s \cap V$ (as Y_s is open in

Y, we can in fact do so with $Y_s \subseteq V$). Then we have

$$X \cap V = X \cap Y \cap V = X \cap Y_s \cap V \subseteq Y_s \cap V$$

So any point of $X \cap Y_s$ has an open neighbourhood in X that is contained in an open neighbourhood of it in Y_s. But we saw that X, Y_s are complex manifolds of dimensions g and d respectively, so $g \leq d$. Recall that we showed that $d \leq g$, so we also get $g = d$. But since Y differed from Y_s by a Zariski-closed set, which is also closed in the affine topology we can shrink V to get $X \cap V = Y_s \cap V$. Thus $X \cap Y_s$ is also open in Y_s. We have shown that it is a nonempty clopen subspace of Y_s, and Y_s is connected. Thus $Y_s \subseteq X$. Thus X also contains the closure of Y_s in Y, which is Y itself. Thus we have $X = Y$, and Y is a projective variety of dimension g as desired. □

4. The Field of Abelian functions on a Polarized Abelian Variety

Definition 4.1. If X is an affine Zariski-closed set in $\mathbb{K}^n$ such that for a subfield K of $\mathbb{K}$ the ideal $I(X)$ has a generating set that lies in $K[X_1, \dots, X_n] \subseteq \mathbb{K}[X_1, \dots, X_n]$, we say that K is a field of definition for X, or that X is defined over K.

We then write $I_K(X) := I(X) \cap K[X_1, \dots, X_n]$

Proposition 4.2. *If X is an affine Zariski-closed set in $\mathbb{K}^n$ defined over a subfield K, then we have an identification*

$$\left({}^{K[X_1,\dots,X_n]}\!/_{I_K(X)} \otimes_K \mathbb{K}\right) \cong {}^{\mathbb{K}[X_1,\dots,X_n]}\!/_{I(X)}$$

Definition 4.3. Assume that X is an affine variety in $\mathbb{K}^n$ defined over a subfield $K \subseteq \mathbb{K}$. Then a point $x = (x_1, \dots, x_n)$ is called generic over K if the homomorphism $K[t_1, \dots, t_n] := {}^{K[T_1,\dots,T_n]}\!/_{I_K(X)} \to K[x_1, \dots, x_n]$ sending t_i to x_i is an isomorphism, where t_i denotes the class of the symbol T_i modulo the ideal $I_K(X)$ and $K[x_1, \dots, x_n]$ is the subring of $\mathbb{K}$ generated by the elements $x_i \in \mathbb{K}$ (the homomorphism is well defined as $x \in X$). This is equivalent to the transcendence degree of $K(x_1, \dots, x_n)$ over K being equal to $\dim(X)$.

Proposition 4.4. *If X is an affine variety in $\mathbb{K}^n$ defined over a subfield $K \subseteq \mathbb{K}$ and the transcendence degree of $\mathbb{K}$ over K is at least $\dim(X)$, then there exists a generic point of X over K.*

Proposition 4.5. *If the algebraically closed field $\mathbb{K}$ is of characteristic zero (and in all cases we consider it will be $\mathbb{C}$) and K is a subfield, a point $(x_1, \dots, x_n) \in \mathbb{K}^n$ is a generic point for a variety defined over K precisely when K is integrally closed in $K(x_1, \dots, x_n)$ (that is every element of $K(x_1, \dots, x_n)$ algebraic over K is an element of K), sometimes also called maximally algebraic.*

Proposition 4.6. *If X is an affine variety in $\mathbb{K}^n$ defined over a subfield $K \subseteq \mathbb{K}$, any generic point x of X over K is also a simple point of X.*

Our primary application of this notion will be in the following scenario. Let X be an affine variety in $\mathbb{C}^n$ and let $P_1, \dots, P_m$ be a finite generating set of $I(X)$. If we simply adjoin all the coefficients of the P_i to $\mathbb{Q}$ to get a subfield K of $\mathbb{C}$, then X is defined over K by construction. Further K is countable (since an image, polynomial ring in one variable, or quotient ring of a countable ring is countable).

As $\mathbb{C}$ is itself uncountable, its transcendence degree over any countable subfield K is infinite. Thus in particular X will always have a generic point over K. However we in fact have the following result. (A discussion of the proof can be found in Igusa III.4 as well)

Theorem 4.7. *Consider a countable subfield K of $\mathbb{C}$. If X is an affine variety in $\mathbb{C}^n$ defined over K, then the set of generic points of X over K is dense in X (with respect to the analytic topology).*

Corollary 4.8. *Let X be an irreducible affine variety in $\mathbb{C}^n$ and let $R := \mathbb{C}[x_1, \ldots, x_n] = \mathbb{C}[T_1, \ldots, T_n]/I(X)$. Let $a = (a_1, \ldots, a_n)$ be a point of X, then $(T_1 - a_1, \ldots, T_n - a_n) = I(a)$ is a maximal ideal containing, $I(X)$, and defines a prime $\mathfrak{p}$ of R. Consider an element $\frac{P}{Q}$ of $F(R)$, and let Y be an affine Zariski-closed set strictly contained in X and also containing all points of X where Q vanishes.*

Assume that $\frac{P(a^n)}{Q(a^n)}$ is bounded for any sequence (a^n) in $X \setminus Y$ (so in particular such a condition makes sense) that converges to a. Then $\frac{P}{Q}$ is integral over $R_{\mathfrak{p}}$.

Proof. Assume for the purpose of contradiction that this is not true. Let $\eta := \frac{Q}{P}$, which exists since then $\frac{P}{Q}$ must be nonzero. Let K be the kernel of $\mathbb{C}[T_1, \ldots, T_n, t] \to \mathbb{C}[x_1, \ldots, x_n, \eta]$ and let $\bar{X} := V(K) \subseteq \mathbb{C}^{n+1}$ be the affine variety it defines. Then $(a, 0) \in \bar{X}$, as otherwise there is a polynomial $f \in R[t]$ such that $f(\eta) = 0$ (so it is represented by a polynomial in the kernel) but $f(0) \notin \mathfrak{p}$ (the polynomial in question does not vanish at $(a, 0)$). However this will imply that $\frac{P}{Q}$ is integral over $R_{\mathfrak{p}}$, a contradiction. Let $\bar{Y}$ be the points of $\bar{X}$ lying over Y, then $\bar{Y}$ is an affine Zariski-closed set strictly contained in $\bar{X}$. Consider a countable subfield $\mathbb{K}$ of $\mathbb{C}$ over which $\bar{X}$ and $\bar{Y}$ are defined, then we may find a sequence of generic points (a^n, b^n) of $\bar{X}$ converging to $(a, 0)$ by the denseness guaranteed by the theorem, necessarily contained in $\bar{X} \setminus \bar{Y}$ since the points are generic for $\bar{X}$. They however lie in the locus defined by $Q - P\eta = 0$, so $Q(a^n) - P(a^n)b^n = 0$. But the (a^n) converge to a and $\frac{P(a^n)}{Q(a^n)} = \frac{1}{b^n}$, which is unbounded. This is a contradiction. ☐

Observation 4.9. *Let X be a projective variety in $\mathbb{P}^{\mathrm{N}}(\mathbb{C})$. We have for each $0 \leq i \leq N$ local charts $\mathbb{C}^N \cong U_i \subseteq \mathbb{P}^{\mathrm{N}}(\mathbb{C})$ where U_i is the set of points with nonzero ith homogenous coordinate. Under this, $X \cap U_i$ corresponds to an affine variety Y (also denoted X_i) in $\mathbb{C}^N$, and the corresponding ideal $I(Y)$ can be described as a "dehomogenisation" of $I(X)$. If $I(X)$ is generated by homogenous polynomials of the variables $X_0, \ldots, X_N$, then $I(Y)$ is generated by the polynomials in the variables $X_0, \ldots, X_{i-1}, X_{i+1}, \ldots X_N$ obtained by substituting $X_i = 1$. (Conversely, $I(X)$ can be obtained by "homogenising" $I(Y)$, by replacing a variable X_j by a quotient $\frac{X_j}{T}$, multiplying by a suitable power of T to clear denominators, and reindexing to get $T = X_i$)*

Definition 4.10. Let Z/L be a polarized abelian variety. Then we call the (global) meromorphic functions on it abelian functions. In the special case that $Z = \mathbb{C}$ is 1-dimensional, we call them elliptic functions.

Recall that for S a graded domain, we denote by $F(S)$ the total quotient field and by $F_0(S)$ the subfield of elements that can be written as quotients of homogenous elements of the same degree.

Theorem 4.11. *Let $S = S(Q, l, \psi)$ be the ring of theta functions of a polarized abelian variety $(Z/L, (kH)_{k>0})$ as before. Then $F_0(S)$ is precisely the field $\mathscr{M}(Z/L)$ of abelian functions.*

Proof. The elements of S are holomorphic, so the inclusion $\mathscr{O}(Z) \subseteq \mathscr{M}(Z)$ induces an embedding $F_0(S) \subseteq \mathscr{M}(Z/L)$ (where we identify $\mathscr{M}(Z/L)$ as the subset of $\mathscr{M}(Z)$ given by the translation invariant meromorphic functions) since the elements are quotients of the same degree (so the automorphy factors cancel) . It suffices to show that every non-zero meromorphic function is represented by an element of $F_0(S)$. Consider such a nonzero element $f = (f_x)_{x \in Z/L} \in \mathscr{M}(Z/L)$.

Each $f_x = \frac{g_x}{h_x}$ for some $g_x, h_x \in \mathscr{O}_{Z/L,x}$, both nonzero as f is a unit. We know from Talks 1-2 that $\mathscr{O}_{Z/L,x}$ is a UFD, so we may even assume that the g_x, h_x are coprime for each x. Locally, we have around every point x a neighbourhood V and $g, h \in \mathscr{O}_{Z/L}(V)$ with h nonzero such that we have $f_x = \frac{g_x}{h_x}$ determined by the stalks of these local sections. We claim that we can construct a such a local representation such that the stalks of the local sections are also coprime at each point.

Consider such a local representation $\{(V^i, g^i, h^i)\}$ (so that the V^i cover Z/L). We will refine this local representation to get one where the stalks are coprime as desired. Consider $(V, g, h) = (V^i, g^i, h^i)$ for some i, then for an arbitrary $x \in V$ let k_x be a greatest common divisor of g_x and h_x, then there is a smaller neighborhood W of x and a section k on W with stalk k_x at x such that k divides both g, h on W. Then we may represent f by $(W, \frac{g}{k}, \frac{h}{k})$ on W as well. Doing so for each point we may cover V by such W's, and doing so for each i get a local representation of f as desired.

So consider such a local representation $\{(V^i, g^i, h^i)\}$ for f. We claim that the $\{(V^i, g^i)\}$ and $\{(V^i, h^i)\}$ define positive divisors. We need only check that on nonempty intersections $V^i \cap V^j$, they are unit multiples of each other. Note that we have $g^i h^j = h^i g^j$, that is stalkwise we have

$$g^i_x h^j_x = h^i_x f_x h^j_x = h^i_x g^j_x$$

But we have ensured that the stalks of the g^i, h^i and g^j, h^j are coprime, so they differ by units stalkwise and hence by units throughout.

Let $\pi \colon Z \to Z/L$ be the projection. We may identify f with $f \circ \pi$, a meromorphic function on Z. By Theorem 3.2 of Talk 3, there is a theta function θ, say of type (Q', l', ψ') such that $\pi^{-1}[\{(V^i, h^i)\}]$ is represented by (Z, θ). So then $f\theta$ is defined by a theta function of type (Q', l', ψ') as well (for instance, the positive divisor $[\{(V^i, h^i)\}]$ corresponds also to a theta function θ', and then θ' and $f\theta$ define the same divisor, so differ by a holomorphic unit. One then compares the transformation formula to conclude).

Let $Q'' := d(Q + Q'), l'' := d(l + l'), \psi'' := (\psi\psi')^d$ for $d \geq 3$. Let $\theta_0, \dots, \theta_N$ and $\theta''_0, \dots, \theta''_{N''}$ denote bases of S_d and $\mathscr{L}_{(Q'', l'', \psi'')}(Z/L)$ respectively, then they both define projective embeddings of Z/L by the results in the previous sections. Let X and X'' respectively denote their images, and let $R := \mathbb{C}[\theta_0, \dots, \theta_N] \cong \mathbb{C}[T_0, \dots, T_N]\big/_{I(X)}$ and $R'' := \mathbb{C}[\theta''_0, \dots, \theta''_{N''}] \cong \mathbb{C}[T_0, \dots, T_{N''}]\big/_{I(X'')}$ denote the corresponding rings of functions. Then,

$$F_0(R) := \mathbb{C}\big(\frac{\theta_1}{\theta_0}, \dots, \frac{\theta_N}{\theta_0}\big), F_0(R'') = \mathbb{C}\big(\frac{\theta''_1}{\theta''_0}, \dots, \frac{\theta''_{N''}}{\theta''_0}\big)$$

as subfields of $\mathscr{M}(Z/L)$.

Note that $S_d \cdot \mathscr{L}_{(dQ', dl', \psi'^d)}(Z/L) \subseteq \mathscr{L}_{(Q'', l'', \psi'')}(Z/L)$. Thus we in fact have an inclusion $F_0(R) \subseteq F_0(R'')$ (which can be seen as $\frac{\theta_i}{\theta_0} = \frac{\theta_i \theta'}{\theta_0 \theta'} \in F_0(R'')$ for some nonzero $\theta' \in \mathscr{L}_{(dQ', dl', \psi'^d)}(Z/L)$). Note that we also have $f = \frac{(f\theta)\theta^{d-1}\theta_0}{\theta^d \theta_0} \in F_0(R'')$.

Lemma 4.12. $F_0(R) = F_0(R'')$

Proof. Consider some nonzero $\theta' \in \mathscr{L}_{(dQ',dl',\psi'^d)}(Z/L)$ and write the products $\theta_i\theta'$ in terms of the basis θ''_j, so $\theta_i\theta' = \sum_{j=1}^{N''} c_{ij}\theta''_j$. Let K be a countable subfield of $\mathbb{C}$ that contains the c_{ij} and the coefficients of a pair of finite homogenous ideal generators for $I(X)$ and $I(X'')$ respectively (for instance, adjoin all these elements to $\mathbb{Q}$). Let Y, Y'' be the affine varieties corresponding to the points of X, X'' with nonzero 0th coordinate (that is, $Y = X_0, Y'' = X''_0$). Note that this corresponds to the image of $\theta_0^{-1}(\mathbb{C}^\times)$ (resp. $\theta_0''^{-1}(\mathbb{C}^\times)$). Then, Y and Y'' are affine varieties defined over K. Let $y'' = (y''_i)$ be a generic point of Y'' over K. Then the isomorphism $K[\frac{\theta''_1}{\theta''_0}, \ldots, \frac{\theta''_{N''}}{\theta''_0}] \to K[y''_0, \ldots, y''_{N''}]$ sending $\frac{\theta''_i}{\theta''_0}$ to y''_i restricts to an isomorphism $K[\frac{\theta_1}{\theta_0}, \ldots, \frac{\theta_N}{\theta_0}] \to K[y_1, \ldots, y_N]$ sending $\frac{\theta_i}{\theta_0}$ to $y_i := \frac{c_{i0}+\sum_{j>0} c_{ij}y''_j}{c_{00}+\sum_{j>0} c_{0j}y''_j}$ (using $\frac{\theta_i}{\theta_0} = \frac{\theta_i\theta'}{\theta_0\theta'}$ and the definition of the c_{ij} in terms of such products).

Thus $K(y_1, \ldots, y_N) \subseteq K(y''_1, \ldots, y''_{N''})$ (by construction) and both have transcendence degree over K ($\dim(X) = \dim(Z/L) = \dim(X'')$). Thus it is a finite algebraic extension. We will show that the two fields are in fact the same. Consider any conjugate K' in $\mathbb{C}$ of $K(y''_1, \ldots, y''_{N''})$ over $K(y_1, \ldots, y_N)$, let y'''_i be the image of y''_i. Then $(y'''_1, \ldots, y'''_{N''})$ is another generic point of Y'' over K. Let z'' and z''' be the preimages of $(y''_1, \ldots, y''_{N''})$ and $(y'''_1, \ldots, y'''_{N''})$ in Z/L, so that

$$y''_j = \frac{\theta''_j(z'')}{\theta''_0(z'')}$$

and

$$y'''_j = \frac{\theta''_j(z''')}{\theta''_0(z''')}$$

for each j. Thus we have for each i

$$\frac{\theta_i(z'')}{\theta_0(z'')} = y_i = \frac{\theta_i(z''')}{\theta_0(z''')}$$

This implies that z'' and z''' have the same image in Y, and hence $z'' = z'''$, thus each $y''_j = y'''_j$ and the conjugate K' must be $K(y''_1, \ldots, y''_{N''})$ itself. Consequently, $K(y_1, \ldots, y_N) = K(y''_1, \ldots, y''_{N''})$, and so each $\frac{\theta''_j}{\theta''_0}$ can be written as a rational function of polynomials in the $\frac{\theta_i}{\theta_0}$ with coefficients in K, so also with coefficients in $\mathbb{C}$. This proves the lemma.

□

Thus, $f \in F_0(R)$. But $R \subseteq S^{(d)}$, so $F_0(R) \subseteq F_0(S^{(d)}) = F_0(S)$ (cf. [Igusa, Lemma 4, Section III.1]), so $f \in F_0(S)$. As f was arbitrary, we have the reverse inclusion as well. □

Corollary 4.13. *S is finitely generated over $\mathbb{C}$.*

Proof. Recall [Igusa, Lemma 8, Section III.1] which asserts that that a graded integral domain S over a characteristic 0 field K such that $S^{(d)}$ is finitely generated over K for some d is itself finitely generated over K. We had also [Igusa, Lemma 10, Section III.2] which asserted that (with S again a graded domain over a characteristic 0 field K) if $x_1, \ldots, x_n$ were nonzero elements of S_d for $d \geq 1$ such that $S^{(d)}$ was integral over $R := K[x_1, \ldots, x_n]$ and $F_0(R) = F_0(S^{(d)})$ then $S^{(d)}$ would be finitely generated over K.

In light of this, it suffices to prove $S^{(d)}$ is finitely generated over $\mathbb{C}$ with S as in the proof of the theorem, as then we would be done by Lemma 8. We showed in the proof of the theorem that $F_0(R) = F_0(S^{(d)})$ where $R := \mathbb{C}[\theta_0, \dots, \theta_N]$, so it suffices to show that $S^{(d)}$ is integral over R.

[Igusa, Lemma 11, Section III.2] implies that to show that a $\theta \in S_{kd}$ for some k is integral over R it suffices to show that $\frac{\theta}{\theta_i^k}$ is integral over $R_i := \mathbb{C}[\frac{\theta_0}{\theta_i}, \dots, \frac{\theta_N}{\theta_i}]$ for each i. Our approach will be to apply the corollary to the denseness of generic points to the affine variety $Y := X_i$ (nonvanishing of the ith coordinate) where X is the image of the embedding determined by the θ_j as in the proof of the theorem (so that in particular $R_i \cong \mathbb{C}[T_1, \dots, T_N] / I(Y)$). So consider a point y of Y and a sequence $(y^n) \to y$. We may take preimages in Z/L, to get a sequence $z^n \to z$. Then $\frac{\theta}{\theta_i^k}(y^n) = \frac{\theta(z^n)}{\theta_i(z^n)^k}$, and this is necessarily bounded as $\theta_i(z) \neq 0$, so all but finitely many points z^n will lie in a closed (hence compact) subspace of Z/L where θ_i is nonzero, hence has a nonzero minimum absolute value (and θ has a maximum absolute value). So from the corollary we see that $\frac{\theta}{\theta_i^k}$ is integral over the localisation of R_i at the the maximal ideal corresponding to y. But y was arbitrary and every maximal ideal corresponds to some y.

The claim then follows as at the end of [Igusa, Section III.2, page 98] it was shown that if an element of the quotient field of a domain is integral over its localisations at every maximal ideal, then it is integral over the domain itself. □

5. Divisors and Line Bundles

Recall that in Talk 3 we associated to any complex manifold X a map

$$\mathscr{M}_X^* / \mathscr{O}_X^* (X) \to H^1(X, \mathscr{O}_X^*)$$

by considering the long exact sequence associated to the short exact sequence

$$0 \to \mathscr{O}_X^* \to \mathscr{M}_X^* \to \mathscr{M}_X^* / \mathscr{O}_X^* \to 0$$

We identified the Cech cohomology group $\check{H}^1(X, \mathscr{O}_X^*)$ with isomorphism classes of holomorphic line bundles (which we called the Picard group, cf. Talk 4) on X, and so this map can be seen (as we have identified Cech cohomology with (derived functor) cohomology) as a map assigning to a divisor an isomorphism class of line bundles. So in this manner we get an injection from the group of line bundles modulo the relation of linear equivalence (since its kernel is the image of $\mathscr{M}_X^*(X)$, and we defined linear equivalence to be the equivalence relation generated by it) to the Picard group.

$$(\mathscr{M}_X^* / \mathscr{O}_X^*)(X) / \sim \;\rightarrowtail H^1(X, \mathscr{O}_X^*)$$

Remark 5.1. It is a result of Algebraic Geometry that if X is a projective variety, then this map is an isomorphism, and we get the classical definition of the Picard group as the group of divisors modulo linear equivalence.

Theorem 5.2. *Let $X = Z/L$ be an abelian variety, then the map defined above is an isomorphism.*

$$(\mathscr{M}_X^* / \mathscr{O}_X^*)(X) / \sim \;\xrightarrow{\simeq} H^1(X, \mathscr{O}_X^*)$$

Proof. It suffices to prove that ${\mathscr{M}^*_X}/{\mathscr{O}^*_X}(X) \to H^1(X, \mathscr{O}^*_X)$ is a surjection. In light of the (subsequence of the) long exact sequence in cohomology

$$\ldots \to \mathscr{M}^*_X(X) \to {\mathscr{M}^*_X}/{\mathscr{O}^*_X}(X) \to H^1(X, \mathscr{O}^*_X) \to H^1(X, \mathscr{M}^*_X) \to \ldots$$

We need only show that the map $H^1(X, \mathscr{O}^*_X) \to H^1(X, \mathscr{M}^*_X)$ is the zero map. This is precisely the map induced on H^1 by the inclusion $\mathscr{O}^*_X \rightarrowtail \mathscr{M}^*_X$, and we may describe it on the level of Cech cohomology.

Recall that when a line bundle $\mathcal{L}$ corresponded to a Cech-cocycle (g_{ij}) with respect to a cover $\mathfrak{U}$ (i.e, represented by a class in $\check{H}^1(\mathfrak{U}, \mathscr{O}^*_X)$), the g_{ij} could be seen as "transition functions" for a local trivialisation of $\mathcal{L}$ with respect to $\mathfrak{U}$. The point to be made is that the property of being a coboundary (boundary of a 0-cocycle) can be seen as the existence of a "holomorphic section", that is local holomorphic functions (so local sections of $\mathcal{L}$ under the trivialisation isomorphisms) that are compatible with the transition functions. But algebraically, the criterion for the image of the cocycle in $\check{H}^1(\mathfrak{U}, \mathscr{M}^*_X)$ to be a coboundary is formally identical.

But in light of the isomorphism between $\check{H}^1(X, \mathscr{O}^*_X)$ and $H^1(Z, \mathscr{O}^*_Z(Z))$ we may state this in terms of a factor of automorphy u for $\mathcal{L}$, we must prove that there is a nonzero meromorphic function $f \in \mathscr{M}^*_Z(Z)$ such that $f(z+l) = f(z)u(l,z)$ for $z \in Z, l \in L$ (call such an f a "meromorphic section" in analogy with the "holomorphic sections", the theta functions).

Let H' be a positive definite Riemann form for the abelian variety X with respect to L. We know from Talk 4 that $\mathcal{L} \cong \mathcal{L}_{(H,\chi)}$ for an Appell-Humbert datum (H, χ), and there exists a semicharacter χ' such that (H', χ') is an Appell-Humbert datum as well.

Our strategy is as follows, we will construct such an f as a ratio of theta functions (this is similar to what we did in the previous section, and we will use a similar approach as well). Our trick is to use the following lemma, which will allow us to construct Appell-Humbert data with positive definite forms.

Lemma 5.3. *Let A and B Hermitian matrices and let A be positive definite. Then there exists an invertible matrix T such that $T^*AT = I$ and T^*BT is a diagonal matrix.*

Proof. This is the statement of [Prasolov-LinAlg, Theorem 20.1] □

Corollary 5.4. *There is an integer k such that in the Appell-Humbert datum*

$$(\tilde{H}, \tilde{\chi}) := (H, \chi) + k(H', \chi') = (H + kH', \chi\chi'^k)$$

the form $\tilde{H}$ is positive definite.

Proof. Translating the statement to one about forms (with $A = H'$, $B = H$), we see that there is a basis of Z (determined by the factor T), such that if we denote the coordinates of points z, y by z_i, y_j with respect to that basis, we have

$$H'(y,z) = \sum_{i=1}^{g} y_i \bar{z}_i$$

$$H(y,z) = \sum_{i=1}^{g} d_i y_i \bar{z}_i$$

for some real $d_1, \ldots, d_g$. Any k whose magnitude is greater than the smallest of the d_i will do the trick. □

Recall that for an Appell-Humbert datum (H, χ) with H positive definite (and hence $A := Im(H)$ is non-degenerate, in fact we have seen that the two notions are equivalent for a Hermetian form H; See [Igusa, Section II.3]) the space $\mathscr{L}_{(H,l)}(X)$ is non-zero (recall from Theorem 1.1 of Talk 6 that for an Appell-Humbert datum (H, χ) this is equivalent to H being positive-semidefinite and χ being strongly associated to $A := Im(H)$, which for instance is immediate when A is non-degenerate.).

So we may take nonzero theta functions $\tilde{\theta} \in \mathscr{L}_{(\tilde{H},\tilde{\chi})}(X)$, $\theta' \in \mathscr{L}_{(kH',\chi'^k)}(X)$. Recall that the factor of automorphy associated to $\mathscr{L}_{(H,\chi)}(X)$ can be written as $u(l,z) = \chi(l)e^{\pi(H(z,l)+\frac{H(l,l)}{2})}$. Set $f := \frac{\tilde{\theta}}{\theta'}$, then for $z \in Z, l \in L$:

$$f(z+l) = \frac{\tilde{\theta}(z+l)}{\theta'(z+l)} = \frac{\tilde{\theta}(z)\tilde{\chi}(l)e^{\pi(\tilde{H}(z,l)+\frac{\tilde{H}(l,l)}{2})}}{\theta'(z)\chi'^k(l)e^{\pi(kH'(z,l)+\frac{kH'(l,l)}{2})}} = f(z)\chi(l)e^{\pi((\tilde{H}-kH')(z,l)+\frac{1}{2}(\tilde{H}-kH')(l,l))}$$
$$= f(z)\chi(l)e^{\pi(H(z,l)+\frac{H(l,l)}{2})}$$

as desired. $\square$

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