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On a characterization of Higher Semiadditivity

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ON A CHARACTERIZATION OF HIGHER SEMIADDITIVITY

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Abstract. In [HopLur13], M. Hopkins and J. Lurie introduce for $m \geqslant -2$, a notion of m-semiadditivity. This generalizes the classical notion of a semiadditive (infinity) category. Intuitively, m-semiadditive infinity categories are those in which limits and colimits of diagrams indexed by m-finite spaces (that is, m-finite infinity groupoids) are canonically equivalent. In [Har20], Y. Harpaz proves a universal property of the infinity category of spans of n-finite spaces with m-truncated wrong way maps. This is used to establish an equivalent characterization of m-semiadditivity in terms of a well behaved, essentially unique action of this category of spans. This has the advantage of not only providing a more succinct method of detecting m-semiadditivity, but also providing a versatile structure to work with m-semiadditive infinity categories. In this thesis, we survey this sequence of results.

Contents

Introduction

A 1-category $\mathcal{C}$ is called semiadditive when it is pointed, and for all finite families of objects $\{X_i\}_{i\in I}$, the map $\coprod_{i\in I} X_i \to \prod_{i\in I} X_i$ "represented by the diagonal matrix"[1] is invertible. Deconstructing this definition, this is asking that:

(1) The category $\mathcal{C}$ is pointed, that is, the map $\varnothing \to *$ is invertible.
(2) For all diagrams in $\mathcal{C}$ indexed by finite discrete categories, the canonical map induced by the diagonal matrix from the colimit to the limit is invertible.

The second condition takes the form of asking that for all diagrams in $\mathcal{C}$ indexed by a specified class of categories, there is a "canonical map" from the colimit to the limit that has to be invertible. The first condition is also of this form, the class of categories here just being the singleton comprising of the initial category.

Remark 0.1. The second condition is not strictly well-defined without the first condition in context, as we need the notion of zero maps to even define the "map induced by the diagonal matrix". However, the first is also an instance of the second, as we may take the finite indexing set to be empty.

So far our indexing categories have all been discrete[2]. We now look for a notion of higher semiadditivity, for which we expand our class of index categories.

Our first pathology occurs when we consider index diagrams with non-invertible morphisms. For instance, if we consider the simplest such example, Δ^1 and a pointed category $\mathcal{C}$, the colimit of an arrow $\Delta^1 \to \mathcal{C}$ is its target, and its limit is its source. The canonical map from the colimit to the limit, which we will henceforth call the "Norm map" will in fact turn out to be just the zero map. Therefore, asking for the norm maps to be equivalences in such cases excludes most of our cases of interest, and we will therefore restrict to indexing diagrams that are infinity groupoids (or equivalently, spaces).

For similar reasons, we will require that our indexing infinity groupoids are π-finite, that is, that they have finitely many components (finite π_0) and that each homotopy group is finite as well. Indeed, even abelian categories and stable infinity categories, the standard examples of semiadditivity have infinite products and coproducts that differ in general.

Finite sets are the prototypical 0-finite spaces[3], that is, infinity groupoids with finitely many connected components and no n-morphisms for nonzero n. One can define in the same manner the notions of pointed and semiadditive infinity categories. Then, as limits and colimits of functors between nerves of 1-categories are computed in the 1-categorical sense, a 1-category is pointed or semiadditive if and only if it is so as an infinity category. We call the property of semiadditivity henceforth 0-semiadditivity, the property of colimits and limits of diagrams indexed by 0-finite spaces coinciding.

The (-1)-finite spaces are the empty space and the contractible ones. As the limit and colimit of a diagram with contractible source can both be computed as the image of any point, they are identified by the identity. Consequently, we can think of being pointed as "(-1)-semiadditivity". Similarly, as the (-2)-finite spaces are the contractible ones, we can say that every infinity category is "(-2)-semiadditive".

The work of Hopkins and Lurie in [HopLur13, Section 4] introduces for each m, a notion of m-semiadditivity. Intuitively, a category is m-semiadditive if for every diagram

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[1] That is, the map $\coprod_{i\in I} X_i \to \prod_{i\in I} X_i$ determined by the composite $X_i \to \coprod_{i\in I} X_i \to \prod_{j\in I} X_j \to X_j$ being the identity if $i = j$ and the zero map otherwise.

[2] As limits and colimits are invariant under equivalences of the indexing diagrams, we can say the same for indexing categories that are disjoint unions of contractible ones.

[3] For every integer $m \geqslant -2$, there is a notion of an m-finite space (recalled later on).

in it indexed by an m-finite space, there is a “Norm map” from its colimit to its limit that is an equivalence.

This notion is developed inductively. What is really provided is for each n, a criterion for an n-semiadditive infinity category to be $(n+1)$-semiadditive. In particular for a diagram indexed by an n-finite space, to even define the norm map that we wish to be an equivalence in the infinity category we consider, it is assumed that the infinity category is $(n-1)$-semiadditive.

In [Har20], Harpaz proves a universal property of a class of infinity category of spans. To be precise, it is shown that the infinity category of spans n-finite spaces with m-truncated wrong way maps is the free m-semiadditive category with colimits indexed by n-finite spaces generated by a point. In particular, the infinity category of spans of n-finite spaces is the free n-semiadditive infinity category, generated by a point.

The Cartesian symmetric monoidal structure on spaces induces a symmetric monoidal structure on the category of spans of n-finite spaces, which can be shown to preserve colimits indexed by n-finite spaces. Harpaz uses the universal property of the infinity category of spans of n-finite spaces to show that the property of n-semiadditivity is equivalent to the existence of a colimit compatible action of the category of spans of n-finite spaces.

The equivalence of these characterizations is useful in practice, while n-semiadditivity is a property of an infinity category, the action of the span category provides a useful structure to work with n-semiadditive infinity categories.

1. Spans in infinity categories

1.1. The definition of a span.

Definitions 1.1. A *span* (or *correspondence*) in an infinity category $\mathcal{C}$ is a diagram of the form

$$X \longleftarrow Z \longrightarrow Y$$

Thinking of infinity categories as quasicategories, a span in an infinity category is a Λ^2_0 shaped diagram (where by Λ^n_k we mean the k-th horn in the standard n-simplex Δ^n).

If we distinguish X as the *source* and Y as the *target*, we call such a diagram a span from X to Y and call the $Z \to X$ map the *wrong way map*. For convenience, we also call the $Z \to Y$ map the *right way map*.

More generally, if $K \subseteq \mathcal{C}_1$ is a class of morphisms (1-simplices) of $\mathcal{C}$, we will call a span in $\mathcal{C}$ a span in $(\mathcal{C}, K)$ if the wrong way map lies in K.

Remark 1.2. The notion of spans is due to Yoneda([Yon54]) and Bénabou([Bén67]). One may recognize spans as being the diagram whose colimit gives pushouts, or as giving the morphisms in a calculus of fractions.

A third application of spans is its use in describing “push-pull” behavior. A prototypical example is that one has for a suitable pair $(\mathcal{C}, \mathcal{C}^\dagger)$ of an infinity category $\mathcal{C}$ and a subcategory $\mathcal{C}^\dagger$, a functor $\mathcal{F}\colon \mathcal{C} \to \mathcal{D}$ for some infinity category $\mathcal{D}$ and a functor $\mathcal{G}\colon \mathcal{C}^{\dagger\mathsf{op}} \to \mathcal{D}$, such that $\mathcal{G}$ and $\mathcal{F}$ agree on objects. One then hopes to construct a functorial assignment taking a span

$$X \xleftarrow{\;g\;} Z \xrightarrow{\;f\;} Y$$

To the morphism

$$\mathcal{G}X \xrightarrow{\mathcal{G}g} \mathcal{G}Z = \mathcal{F}Z \xrightarrow{\mathcal{F}f} \mathcal{F}Y$$

We will work extensively with an instance of such a construction in our characterization of semiadditivity.

1.2. **Constructing an infinity category of spans.** We would like to define an infinity category of spans in $(\mathcal{C}, K)$. More precisely, we want an infinity category with objects that of $\mathcal{C}$ but 1-morphisms from an object X to an object Y to be given by spans from X to Y in $(\mathcal{C}, K)$. The content of this section is primarily taken from [Har20, §2.1]. The definitions are as in [Bar13].

Remark 1.3. We will work primarily with the model of quasicategories, despite trying to remain stylistically model agnostic. However, a subcategory K of a quasicategory $\mathcal{C}$ is a genuine sub-simplicial set. It will be convenient for us to work with an equivalence invariant notion, so we recall the following definition.

Definitions 1.4. Call a morphism $f\colon x \to y$ in an infinity category $\mathcal{C}$ is a *monomorphism* when the canonical commutative square

$$\begin{array}{ccc} X & \overset{1_X}{=\!=\!=} & X \\ {\scriptstyle 1_X}\Vert & \lrcorner & \downarrow{\scriptstyle f} \\ X & \xrightarrow[f]{} & Y \end{array}$$

is cartesian.

A functor $\mathcal{F}\colon \mathcal{C} \to \mathcal{D}$ between infinity categories is *faithful* when for each pair of objects X, Y of $\mathcal{C}$, the induced map $\mathcal{C}(X, Y) \to \mathcal{D}(\mathcal{F}X, \mathcal{F}Y)$ is a monomorphism in a (suitably large) infinity category of spaces. In other words, every (homotopy) fiber of $\mathcal{C}(X, Y) \to \mathcal{D}(\mathcal{F}X, \mathcal{F}Y)$ is contractible (intuitively, every morphism $\mathcal{F}X \to \mathcal{F}Y$ has an essentially unique preimage) or empty (Every morphism $\mathcal{F}X \to \mathcal{F}Y$ has no preimage).

If there is a faithful functor $\mathcal{F}\colon \mathcal{C} \to \mathcal{D}$, we call $\mathcal{C}$ a subcategory of $\mathcal{D}$ and perform an abuse of notation by writing $\mathcal{C} \subseteq \mathcal{D}$ or $\mathcal{C} \rightarrowtail \mathcal{D}$ (leaving $\mathcal{F}$ implicit).

Remarks 1.5. It can be shown that a $\mathcal{C} \to \mathcal{D}$ is faithful if and only if it defines an equivalence of $\mathcal{C}$ onto a subcategory on the nose (in the sense of being a sub-simplicial set) of $\mathcal{D}$.

In fact, it can also be shown that a $\mathcal{F}\colon \mathcal{C} \to \mathcal{D}$ is faithful precisely when the induced functor $\mathsf{ho}\,\mathcal{F}\colon \mathsf{ho}\,\mathcal{C} \to \mathsf{ho}\,\mathcal{D}$ on homotopy 1-categories is faithful, and the commutative square

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\mathcal{F}} & \mathcal{D} \\ \downarrow & \lrcorner\, h & \downarrow \\ \mathsf{ho}\,\mathcal{C} & \xrightarrow[\mathsf{ho}\,\mathcal{F}]{} & \mathsf{ho}\,\mathcal{D} \end{array}$$

is homotopy cartesian (where we have performed the standard abuse of notation of writing $\mathcal{A}$ for the infinity category which is a nerve of a 1-category $\mathcal{A}$).

Definition 1.6. A subcategory $\mathcal{C} \subseteq \mathcal{D}$ is called *wide* when the inclusion induces an equivalence $\mathcal{C}^{\simeq} \simeq \mathcal{D}^{\simeq}$ on maximal sub-groupoids.

A morphism f of $\mathcal{D}$ is then said to belong to $\mathcal{C}$ (denoted $f : \mathcal{C}$) if it is isomorphic in the arrow category $\mathsf{Fun}(\Delta^1, \mathcal{D})$ to a morphism in (the image of) $\mathsf{Fun}(\Delta^1, \mathcal{C})$.

Notation 1.7. Given an infinity category $\mathcal{C}$ and a wide subcategory $\mathcal{A}$, we call a span in $\mathcal{C}$ a $(\mathcal{C}, \mathcal{A})$-span if the wrong way map belongs to $\mathcal{A}$.

The composition of a pair of spans in an infinity category $\mathcal{C}$ from X to Y and Y to Z

$$\begin{array}{ccccccc} & & A & & & B & \\ & \swarrow & & \searrow & \swarrow & & \searrow \\ X & & & Y & & & Z \end{array}$$

is given by forming the pullback $C := A \times_Y B$ and taking the outermost span in

$$\begin{array}{ccccccc} & & & C & & & \\ & & \swarrow & \lrcorner & \searrow & & \\ & A & & & & B & \\ \swarrow & & \searrow & & \swarrow & & \searrow \\ X & & & Y & & & Z \end{array}$$

We will define an infinity category $\mathsf{Span}(\mathcal{C}, K)$ such that the composition of 1-morphisms takes place in precisely this manner, and will require that the wrong way map in the composite also belongs to K. The following definition ensures this.

Definition 1.8. A *weak CoWaldhausen* structure on an infinity category $\mathcal{C}$ is a wide subcategory $\mathcal{C}^\dagger$, such that any diagram in $\mathcal{C}$

$$\begin{array}{ccc} & & X \\ & & \downarrow{\scriptstyle f} \\ B & \xrightarrow{v} & Y \end{array}$$

where the map f belongs to $\mathcal{C}^\dagger$, fits into a pullback square

$$\begin{array}{ccc} A & \xrightarrow{u} & X \\ {\scriptstyle g}\downarrow & \lrcorner & \downarrow{\scriptstyle f} \\ B & \xrightarrow{v} & Y \end{array}$$

in $\mathcal{C}$, such that the map g belongs to $\mathcal{C}^\dagger$ as well.

A *weak CoWaldhausen infinity category* is a pair $(\mathcal{C}, \mathcal{C}^\dagger)$, of an infinity category $\mathcal{C}$ and a weak CoWaldhausen structure $\mathcal{C}^\dagger$ on $\mathcal{C}$.

Remark 1.9. The property of a wide subcategory $\mathcal{C}^\dagger$ of $\mathcal{C}$ being a weak CoWaldhausen structure guarantees that every pair of spans in $(\mathcal{C}, \mathcal{C}^\dagger)$ such that the target of the first is the source of the second has a "composition" in $(\mathcal{C}, \mathcal{C}^\dagger)$ in the aforementioned sense. Indeed, given a weak CoWaldhausen infinity category $(\mathcal{C}, \mathcal{C}^\dagger)$, one can define an infinity category of spans in $(\mathcal{C}, \mathcal{C}^\dagger)$ which satisfies the properties we expect.

Examples 1.10. For any infinity category $\mathcal{C}$, the maximal sub-groupoid $\mathcal{C}^\simeq$ is the minimal weak CoWaldhausen structure on $\mathcal{C}$.

When $\mathcal{C}$ has pullbacks, then $\mathcal{C}$ itself defines the maximal weak CoWaldhausen structure on $\mathcal{C}$.

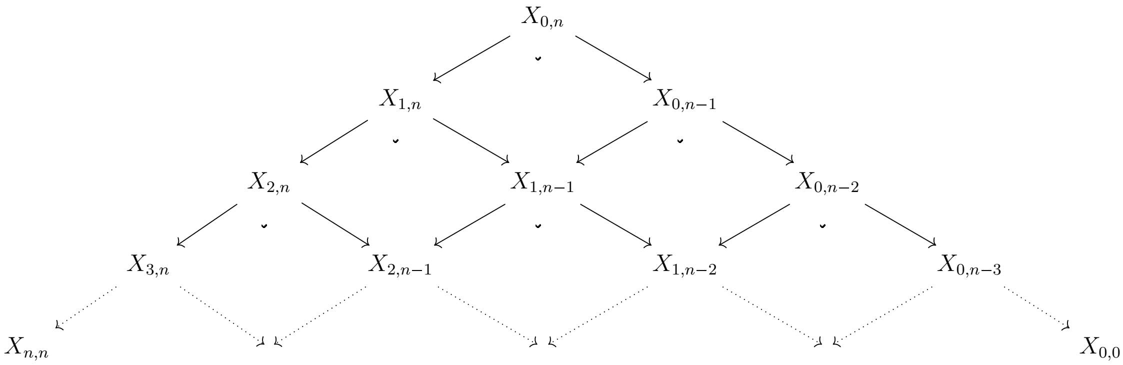
Construction 1.11. Given an infinity category $\mathcal{C}$ and a subcategory E of $\mathsf{ho}\,\mathcal{C}$, we construct a subcategory $\mathcal{C}_E$ of $\mathcal{C}$ by forming the pullback

$$\begin{array}{ccc} \mathcal{C}_E & \rightarrowtail & \mathcal{C} \\ \downarrow & \lrcorner & \downarrow \\ E & \rightarrowtail & \mathsf{ho}\,\mathcal{C} \end{array}$$

Let $F \subseteq \mathcal{C}_1$ be a "class of fibrations", that is a class of 1-morphisms containing all equivalences, closed under compositions, such that pullbacks along morphisms in F exist in $\mathcal{C}$ and further, F is closed under pullbacks. Then applying this construction to the subcategory of $\mathsf{ho}\,\mathcal{C}$ defined by F (necessarily containing all objects) produces a CoWaldhausen structure on $\mathcal{C}$. Our primary examples will be constructed in this manner.

Notation 1.12. When $A \subseteq \mathcal{C}_0$ is a family of objects of an infinity category $\mathcal{C}$, we will denote by $\mathcal{C}_A$ the full subcategory of $\mathcal{C}$ obtained by applying the above Construction 1.11 to the full subcategory of $\mathsf{ho}\,\mathcal{C}$ spanned by A.

We will construct our desired $\mathsf{Span}(\mathcal{C}, \mathcal{C}^\dagger)$ as a quasicategory. We have already decided that the 0-simplices are the objects of $\mathcal{C}$ and the 1-simplices are given by spans in $\mathcal{C}$ whose wrong way maps belong to $\mathcal{C}^\dagger$. Intuitively, an n-simplex in a quasicategory is the data of a composition of the n morphisms determined by the restriction to the spine. Recalling the manner in which spans compose, an n-simplex of $\mathsf{Span}(\mathcal{C}, \mathcal{C}^\dagger)$ should correspond to a diagram of pullbacks in $\mathcal{C}$ of the form



such that all the wrong way maps (the ones of form $X_{i,j} \to X_{i+1,j}$) all belong to $\mathcal{C}^\dagger$. We will proceed to turn this into a definition.

Definition 1.13. Recall that for an infinity category $\mathcal{C}$, the *Twisted Arrow Category* is the infinity category $\mathsf{Tw}(\mathcal{C})$ whose n-simplices are given by maps $\Delta^{n^{\mathsf{op}}} \star \Delta^n \to \mathcal{C}$ of simplicial sets, and the action of simplicial operators given by pre-composition. This is functorial in $\mathcal{C}$ by post-composition.

Furthermore, if $\mathcal{C}$ is (the nerve of) a 1-category, then $\mathsf{Tw}(\mathcal{C})$ is the 1-category with objects given by 1-simplices of $\mathcal{C}$ and morphisms $f \to g$ given by commutative diagrams

$$\begin{array}{ccc} a & \xrightarrow{f} & b \\ {\scriptstyle u}\uparrow & & \downarrow{\scriptstyle v} \\ c & \xrightarrow[g]{} & d \end{array}$$

in $\mathcal{C}$. In particular, $\mathsf{Tw}(\Delta^n)$ is the poset with elements $\{(i,j) \in [n] \times [n] \mid i \leqslant j\}$ and order relation $(i,j) \leqslant (i',j')$ if and only if $i' \leqslant i \leqslant j \leqslant j'$.

Warning 1.14. Some authors call this construction the *Twisted Diagonal*, and call its opposite the Twisted Arrow Category. Despite the fact that we will in fact in our construction ourselves use the opposite $\mathsf{Tw}(\mathcal{C})^{\mathsf{op}}$, we state the definitions as such to remain consistent with [Har20] and [Bar13].

Definition 1.15. For a weak CoWaldhausen infinity category $(\mathcal{C}, \mathcal{C}^\dagger)$, we call a map of simplicial sets $f\colon \mathsf{Tw}(\Delta^n)^{\mathsf{op}} \to \mathcal{C}$ *Cartesian* if each morphism $(i,j) \to (i',j')$ in $\mathsf{Tw}(\Delta^n)$

(that is, whenever $i' \leqslant i \leqslant j \leqslant j'$) induces a Cartesian square

$$\begin{array}{ccc} f(i',j') & \longrightarrow & f(i',j) \\ \downarrow & \lrcorner & \downarrow \\ f(i,j') & \longrightarrow & f(i,j) \end{array}$$

such that the vertical (dashed) maps belong to $\mathcal{C}^\dagger$.

Let $\mathsf{Fun}(\mathsf{Tw}(\Delta^n)^{\mathsf{op}}, \mathcal{C})_{\square,\dagger}$ be the full subcategory of $\mathsf{Fun}(\mathsf{Tw}(\Delta^n)^{\mathsf{op}}, \mathcal{C})$ spanned by the Cartesian maps.

Remark 1.16. As a map out of (the nerve of) a poset is determined by its restriction to the finite totally ordered subsets, a Cartesian $\mathsf{Tw}(\Delta^n)^{\mathsf{op}} \to \mathcal{C}$ corresponds to a "pyramid of pullbacks" as in the discussion on the n-simplices of $\mathsf{Span}(\mathcal{C}, \mathcal{C}^\dagger)$. The higher simplices of $\mathsf{Tw}(\Delta^n)$ encode the remaining coherence data.

Definitions 1.17. For a weak CoWaldhausen infinity category $(\mathcal{C}, \mathcal{C}^\dagger)$, let $Q_*(\mathcal{C}, \mathcal{C}^\dagger)$ be the simplicial simplicial set (simplicial set valued presheaf of Δ) given by the assignment

$$n \mapsto \mathsf{Fun}(\mathsf{Tw}(\Delta^n)^{\mathsf{op}}, \mathcal{C})^{\simeq}_{\square,\dagger}$$

Let $\mathsf{Span}(\mathcal{C}, \mathcal{C}^\dagger)$ denote the simplicial set obtained by taking the vertices at each level.

Observation 1.18. $\mathsf{Span}(\mathcal{C}, \mathcal{C}^\dagger)$ *can be described as the simplicial set whose n-simplices are the Cartesian maps* $\mathsf{Tw}(\Delta^n)^{\mathsf{op}} \to \mathcal{C}$*, and the simplicial operators act by pre-composition.*

Theorem 1.19. *(Barwick; cf.* [Bar13, Proposition 3.4 - Definition 3.8]*) Let* $(\mathcal{C}, \mathcal{C}^\dagger)$ *be a weak CoWaldhausen infinity category . Then* $Q_*(\mathcal{C}, \mathcal{C}^\dagger)$ *is a complete segal space. Consequently, the simplicial set* $\mathsf{Span}(\mathcal{C}, \mathcal{C}^\dagger)$ *is a quasicategory, the infinity category of* $(\mathcal{C}, \mathcal{C}^\dagger)$*-spans.*

Observation 1.20. *We have (unique) isomorphisms* $\mathsf{Tw}(\Delta^0) \cong \Delta^0$ *and*

$$\mathsf{Tw}(\Delta^1) \cong N((0,0) \to (0,1) \leftarrow (1,1)) \cong \Lambda^2_2$$

Consequently, $\mathsf{Span}(\mathcal{C}, \mathcal{C}^\dagger)$ *has objects given by the objects of* $\mathcal{C}$ *and 1-morphisms given by* $(\mathcal{C}, \mathcal{C}^\dagger)$*-spans, as desired.*

Notation 1.21. For an infinity category $\mathcal{C}$ and objects x, y of $\mathcal{C}$, we denote by $\mathsf{Map}^R_{\mathcal{C}}(x,y)$ the fiber

$$\begin{array}{ccc} \mathsf{Map}^R_{\mathcal{C}}(x,y) & \rightarrowtail & \mathcal{C}_{/y} \\ \downarrow & \lrcorner & \downarrow l \\ * & \overset{x}{\rightarrowtail} & \mathcal{C} \end{array}$$

It is a Kan-complex as pullbacks of left fibrations are left fibrations, and left fibrations with target infinity groupoids are Kan fibrations (cf. [Cis19, Proposition 3.5.5]).

There is a canonical equivalence of infinity groupoids (cf. [Cis19, Corollary 5.6.14])

$$\mathsf{Map}^R_{\mathcal{C}}(x,y) \simeq \mathsf{Hom}_{\mathcal{C}}(x,y)$$

Proposition 1.22. *Given a sequence* $\tilde{\mathcal{C}} \subseteq \mathcal{C}^\dagger \subseteq \mathcal{C}$ *of weak CoWaldhausen structures on an infinity category* $\mathcal{C}$ *with finite products, the mapping space* $\mathsf{Span}(\mathcal{C}, \tilde{\mathcal{C}})(X,Y)$ *is equivalent to the full subcategory of* $\mathsf{Span}(\mathcal{C}, \mathcal{C}^\dagger)(X,Y)$ *consisting of the spans whose wrong way maps belong to* $\tilde{\mathcal{C}}$*.*

Proof. Recall that the quasicategories $\mathsf{Span}(\mathcal{C}, \tilde{\mathcal{C}})$ and $\mathsf{Span}(\mathcal{C}, \mathcal{C}^\dagger)$ underlie the complete segal spaces $Q_*(\mathcal{C}, \tilde{\mathcal{C}})$ and $Q_*(\mathcal{C}, \mathcal{C}^\dagger)$. We have full subcategory inclusions

$$Q_n(\mathcal{C}, \tilde{\mathcal{C}}) \subseteq Q_n(\mathcal{C}, \mathcal{C}^\dagger) \subseteq \mathsf{Fun}(\mathsf{Tw}(\Delta^n)^{\mathsf{op}}, \mathcal{C})^{\mathsf{op}}_{\square}$$

given by taking full subcategories spanned by diagrams with wrong way maps belonging to $\tilde{\mathcal{C}}$ (resp. $\mathcal{C}^\dagger$). We will show that the map $\mathsf{Span}(\mathcal{C}, \tilde{\mathcal{C}}) \rightarrowtail \mathsf{Span}(\mathcal{C}, \mathcal{C}^\dagger)$ is a monomorphism of infinity categories.

Lemma 1.23. *Let A and B be quasicategories underlying complete segal spaces A_* and B_*. Then, a morphism $A_* \to B_*$ is a monomorphism of complete segal spaces if and only if the induced $A \to B$ is a monomorphism of infinity categories.*

Proof. A morphism $A_* \to B_*$ is a monomorphism if and only if for each level n we have a Cartesian square of spaces

$$\begin{array}{ccc} A_n & = & A_n \\ \| & \lrcorner & \downarrow \\ A_n & \longrightarrow & B_n \end{array}$$

Thus, one essentially has a Cartesian square on passing to vertices and hence a similar Cartesian square of infinity categories

$$\begin{array}{ccc} A & = & A \\ \| & \lrcorner & \downarrow \\ A & \longrightarrow & B \end{array}$$

Conversely, if $A \to B$ is a monomorphism, we have on exponentiating and taking maximal subgroupoids for each n a Cartesian square

$$\begin{array}{ccc} \mathsf{Fun}(\Delta^n, A)^{\simeq} & = & \mathsf{Fun}(\Delta^n, A)^{\simeq} \\ \| & \lrcorner & \downarrow \\ \mathsf{Fun}(\Delta^n, A)^{\simeq} & \longrightarrow & \mathsf{Fun}(\Delta^n, B)^{\simeq} \end{array}$$

For a complete segal space X_* corresponding to the quasicategory X, we have canonical identifications $X_n \simeq \mathsf{Fun}(\Delta^n, X)^{\simeq}$. Thus the above Cartesian square implies that for each n,

$$\begin{array}{ccc} A_n & = & A_n \\ \| & \lrcorner & \downarrow \\ A_n & \longrightarrow & B_n \end{array}$$

is cartesian, and hence $A_* \to B_*$ is Cartesian, as desired. □

As the inclusions $Q_n(\mathcal{C}, \tilde{\mathcal{C}}) \subseteq Q_n(\mathcal{C}, \mathcal{C}^\dagger)$ of full subcategories are monomorphisms, it thus follows from the lemma that $\mathsf{Span}(\mathcal{C}, \tilde{\mathcal{C}}) \to \mathsf{Span}(\mathcal{C}, \mathcal{C}^\dagger)$ is a monomorphism. We thus have a Cartesian square of infinity categories

$$\begin{array}{ccc} \mathsf{Span}(\mathcal{C}, \tilde{\mathcal{C}}) & = & \mathsf{Span}(\mathcal{C}, \tilde{\mathcal{C}}) \\ \| & \lrcorner & \downarrow \\ \mathsf{Span}(\mathcal{C}, \tilde{\mathcal{C}}) & \longrightarrow & \mathsf{Span}(\mathcal{C}, \mathcal{C}^\dagger) \end{array}$$

This is compatible with exponentiation and fibers. In particular on exponentiating by Δ^1 and taking fibers over (a, b), one gets a Cartesian square of mapping spaces

$$\begin{array}{ccc} \mathsf{Span}(\mathcal{C},\tilde{\mathcal{C}})(X,Y) & = & \mathsf{Span}(\mathcal{C},\tilde{\mathcal{C}})(X,Y) \\ \| & \lrcorner & \downarrow \\ \mathsf{Span}(\mathcal{C},\tilde{\mathcal{C}})(X,Y) & \longrightarrow & \mathsf{Span}(\mathcal{C},\mathcal{C}^{\dagger})(X,Y) \end{array}$$

So $\mathsf{Span}(\mathcal{C},\tilde{\mathcal{C}})(X,Y) \to \mathsf{Span}(\mathcal{C},\mathcal{C}^{\dagger})(X,Y)$ is a monomorphism in spaces. This means that the map is essentially an inclusion of connected components, that is we have a Cartesian

$$\begin{array}{ccc} \mathsf{Span}(\mathcal{C},\tilde{\mathcal{C}})(X,Y) & \rightarrowtail & \mathsf{Span}(\mathcal{C},\mathcal{C}^{\dagger})(X,Y) \\ \downarrow & \lrcorner & \downarrow \\ \pi_0\mathsf{Span}(\mathcal{C},\tilde{\mathcal{C}})(X,Y) & \rightarrowtail & \pi_0\mathsf{Span}(\mathcal{C},\mathcal{C}^{\dagger})(X,Y) \end{array}$$

In our case, the connected components in question are those whose wrong way maps belong to $\tilde{\mathcal{C}}$, so $\mathsf{Span}(\mathcal{C},\mathcal{C}^{\dagger})(X,Y)$ is precisely the claimed full subcategory of $\mathsf{Span}(\mathcal{C},\mathcal{C}^{\dagger})(X,Y)$. ☐

Corollary 1.24. *Given a sequence $\tilde{\mathcal{C}} \subseteq \mathcal{C}^{\dagger} \subseteq \mathcal{C}$ of weak CoWaldhausen structures on an infinity category $\mathcal{C}$ with finite products, the canonical functor $\mathsf{Span}(\mathcal{C},\tilde{\mathcal{C}}) \rightarrowtail \mathsf{Span}(\mathcal{C},\mathcal{C}^{\dagger})$ is a subcategory inclusion (as simplicial sets, even).*

In particular, $\mathcal{C} \simeq \mathsf{Span}(\mathcal{C},\mathcal{C}^{\simeq})$ can be seen as a subcategory of $\mathsf{Span}(\mathcal{C},\mathcal{C}^{\dagger})$ for any weak CoWaldhausen infinity category $(\mathcal{C},\mathcal{C}^{\dagger})$.

Proof. The maps act as the identity on objects, so it suffices to prove that the induced functor on mapping spaces is a full subcategory inclusion. This is however just the conclusion of Proposition 1.22. ☐

Definition 1.25. Consider a weak CoWaldhausen infinity category $(\mathcal{C},\mathcal{C}^{\dagger})$ and a span λ

$$\begin{array}{ccc} & Z & \\ {\scriptstyle g}\swarrow & & \searrow{\scriptstyle f} \\ X & & Y \end{array}$$

from X to Y whose legs both belong to $\mathcal{C}^{\dagger}$. Then we define its *dual*, denoted $\widehat{\lambda}$ to be the span

$$\begin{array}{ccc} & Z & \\ {\scriptstyle f}\swarrow & & \searrow{\scriptstyle g} \\ Y & & X \end{array}$$

Proposition 1.26. *Consider a weak CoWaldhausen infinity category $(\mathcal{C},\mathcal{C}^{\dagger})$. Then for a space K, there is a weak CoWaldhausen structure $\left(\mathcal{C}^K\right)^{\dagger}$ on $\mathcal{C}^K$ consisting of morphisms of functors that are object-wise in $\mathcal{C}^{\dagger}$. Further, we have a canonical identification*

$$\mathsf{Span}(\mathcal{C},\mathcal{C}^{\dagger})^K \simeq \mathsf{Span}(\mathcal{C}^K, \left(\mathcal{C}^K\right)^{\dagger})$$

Proof. Let $(\mathcal{C},\mathcal{C}^{\dagger})$ be a weak CoWaldhausen infinity category and K an infinity groupoid. Let $\left(\mathcal{C}^K\right)^{\dagger} \subseteq \left(\mathcal{C}^K\right)_1$ be the subset of morphisms of functors $K \times \Delta^1 \to \mathcal{C}$ such that the component at each object of K lies in $\mathcal{C}^{\dagger}$.

Consider a cospan

$$\begin{array}{ccc} & & X \\ & & \big\downarrow{\scriptstyle :(\mathcal{C}^K)^{\dagger}} \\ B & \longrightarrow & Y \end{array}$$

such that the map $X \to Y$ belongs to $\left(\mathcal{C}^K\right)^{\dagger}$. For each object of k, there is a Cartesian

$$\begin{array}{ccc} A_k & \longrightarrow & X_K \\ \big\downarrow{\scriptstyle :\mathcal{C}^{\dagger}} \lrcorner & & \big\downarrow{\scriptstyle :\mathcal{C}^{\dagger}} \\ B_k & \longrightarrow & Y_K \end{array}$$

in $\mathcal{C}$ such that the pullback $A_k \to B_k$ belongs to $\mathcal{C}^{\dagger}$. These pullbacks assemble into a Cartesian diagram in $\mathcal{C}^K$

$$\begin{array}{ccc} A & \longrightarrow & X \\ \big\downarrow{\scriptstyle :(\mathcal{C}^K)^{\dagger}} \lrcorner & & \big\downarrow{\scriptstyle :(\mathcal{C}^K)^{\dagger}} \\ B & \longrightarrow & Y \end{array}$$

and hence : $\left(\mathcal{C}^K\right)^{\dagger}$ is indeed a weak CoWaldhausen structure on $\mathcal{C}^K$.

Now, the complete segal space $Q_*(\mathcal{C}^K, \left(\mathcal{C}^K\right)^{\dagger})$ corresponding to $\mathsf{Span}(\mathcal{C}^K, \left(\mathcal{C}^K\right)^{\dagger})$ is such that $Q_n(\mathcal{C}^K, \left(\mathcal{C}^K\right)^{\dagger})$ is the full subcategory of $\mathsf{Fun}(\mathsf{Tw}(\Delta^n)^{\mathsf{op}}, \mathcal{C}^K)^{\simeq}$ spanned by the Cartesian morphisms. In light of the identification $\mathsf{Fun}(\mathsf{Tw}(\Delta^n)^{\mathsf{op}}, \mathcal{C}^K) \cong \mathsf{Fun}(K, \mathsf{Fun}(\mathsf{Tw}(\Delta^n)^{\mathsf{op}}, \mathcal{C}))$, it corresponds to the full subcategory of $\mathsf{Fun}(K, \mathsf{Fun}(\mathsf{Tw}(\Delta^n)^{\mathsf{op}}, \mathcal{C}))^{\simeq}$ of functors that are object-wise Cartesian with respect to $(\mathcal{C}, \mathcal{C}^{\dagger})$. That is,

$$Q_n(\mathcal{C}^K, \left(\mathcal{C}^K\right)^{\dagger}) \cong \mathsf{Fun}(K, \mathsf{Fun}(\mathsf{Tw}(\Delta^n)^{\mathsf{op}}, \mathcal{C})_{\square,\dagger})^{\simeq} = \mathsf{Fun}(K, Q_n(\mathcal{C}, \mathcal{C}^{\dagger}))^{\simeq}$$

(the last equality being a consequence of the fact that if K is an infinity groupoid, $\mathsf{Fun}(K, X)^{\simeq} = \mathsf{Fun}(K, X^{\simeq})^{\simeq}$)

We have canonical equivalences $\mathsf{Fun}(\Delta^n, \mathsf{Span}(\mathcal{C}, \mathcal{C}^{\dagger}))^{\simeq} \simeq Q_n(\mathcal{C}, \mathcal{C}^{\dagger}) = \mathsf{Fun}(\mathsf{Tw}(\Delta^n)^{\mathsf{op}}, \mathcal{C})^{\simeq}_{\square,\dagger}$ for each n, so canonical infinity categorical equivalences

$$\mathsf{Fun}(K, \mathsf{Fun}(\mathsf{Tw}(\Delta^n)^{\mathsf{op}}, \mathcal{C})_{\square,\dagger})^{\simeq} = \mathsf{Fun}(K, \mathsf{Fun}(\mathsf{Tw}(\Delta^n)^{\mathsf{op}}, \mathcal{C})^{\simeq}_{\square,\dagger})^{\simeq} \simeq \mathsf{Fun}(K, \mathsf{Fun}(\Delta^n, \mathsf{Span}(\mathcal{C}, \mathcal{C}^{\dagger}))^{\simeq})^{\simeq}$$

This can be further rewritten as

$$\mathsf{Fun}(K, \mathsf{Fun}(\Delta^n, \mathsf{Span}(\mathcal{C}, \mathcal{C}^{\dagger}))^{\simeq})^{\simeq} = \mathsf{Fun}(K, \mathsf{Fun}(\Delta^n, \mathsf{Span}(\mathcal{C}, \mathcal{C}^{\dagger})))^{\simeq} \cong \mathsf{Fun}(\Delta^n, \mathsf{Fun}(K, \mathsf{Span}(\mathcal{C}, \mathcal{C}^{\dagger})))^{\simeq}$$

As $Q_n(\mathcal{C}^K, \left(\mathcal{C}^K\right)^{\dagger}) \simeq \mathsf{Fun}(\Delta^n, \mathsf{Span}(\mathcal{C}^K, \left(\mathcal{C}^K\right)^{\dagger}))^{\simeq}$ canonically as well, we have functorial identifications

$$\mathsf{Fun}(\Delta^n, \mathsf{Span}(\mathcal{C}^K, \left(\mathcal{C}^K\right)^{\dagger}))^{\simeq} \simeq \mathsf{Fun}(\Delta^n, \mathsf{Fun}(K, \mathsf{Span}(\mathcal{C}, \mathcal{C}^{\dagger})))^{\simeq}$$

that is

$$\mathsf{Hom}_{\mathsf{Cat}_\infty}(\Delta^n, \mathsf{Span}(\mathcal{C}^K, \left(\mathcal{C}^K\right)^{\dagger})) \simeq \mathsf{Hom}_{\mathsf{Cat}_\infty}(\Delta^n, \mathsf{Fun}(K, \mathsf{Span}(\mathcal{C}, \mathcal{C}^{\dagger})))$$

functorially in n. It follows that we have indeed an equivalence

$$\mathsf{Span}(\mathcal{C}^K, \left(\mathcal{C}^K\right)^{\dagger}) \simeq \mathsf{Fun}(K, \mathsf{Span}(\mathcal{C}, \mathcal{C}^{\dagger}))$$

as desired. □

Proposition 1.27. *Consider a weak CoWaldhausen infinity category* $(\mathcal{C}, \mathcal{C}^{\dagger})$ *and objects* X, Y *of* $\mathcal{C}$, *such that* $\mathcal{C}$ *has finite products. There is a canonical identification of* $\mathsf{Hom}_{\mathsf{Span}(\mathcal{C}, \mathcal{C}^{\dagger})}(X, Y)$ *with the full subcategory of* $\left(\mathcal{C}_{/(X \times Y)}\right)^{\simeq}$ *spanned by maps* $Z \to X \times Y$ *such that the projection to* X *belongs to* $\mathcal{C}^{\dagger}$.

Notation 1.28. For an infinity category $\mathcal{C}$ and an object X, we denote by $\mathcal{C}_{/\!\!/X}$ the *fat slice* or *alternative slice*, the fiber of $\mathsf{ev}\colon \mathsf{Fun}(\Delta^1,\mathcal{C}) \to \mathcal{C}$ at X. It is canonically equivalent to the so called regular slice $\mathcal{C}_{/X}$ as infinity categories over X.

Proof. Denote by $\left(\mathcal{C}_{/(X\times Y)}\right)_{\ddagger}^{\simeq}$ the aforementioned full subcategory, and by $\left(\mathcal{C}_{/X}\right)_{\dagger}$ the full subcategory of $\mathcal{C}_{/X}$ spanned by the morphisms $Z \to X$ that belong to $\mathcal{C}^{\dagger}$. Then, the canonical equivalence of infinity categories $\mathcal{C}_{/(X\times Y)} \simeq \mathcal{C}_{/X} \times_{\mathcal{C}} \mathcal{C}_{/Y}$ restricts to

$$\left(\mathcal{C}_{/(X\times Y)}\right)_{\ddagger}^{\simeq} \simeq \left(\mathcal{C}_{/X}\right)_{\dagger}^{\simeq} \times_{\mathcal{C}^{\simeq}} \mathcal{C}_{/Y}^{\simeq}$$

The equivalence $\mathcal{C}_{/X} \simeq \mathcal{C}_{/\!\!/X}$ restricts to one of $\left(\mathcal{C}_{/X}\right)_{\dagger}$ onto the similarly defined $\left(\mathcal{C}_{/\!\!/X}\right)_{\dagger}$. To summarize, we have a canonical equivalence

$$\left(\mathcal{C}_{/(X\times Y)}\right)_{\ddagger}^{\simeq} \simeq \left(\mathcal{C}_{/\!\!/X}\right)_{\dagger}^{\simeq} \times_{\mathcal{C}^{\simeq}} \mathcal{C}_{/\!\!/Y}^{\simeq}$$

Furthermore, the Cartesian squares (in particular fiber sequences) defining the fat slices assemble into a Cartesian square

$$\begin{array}{ccc}
\left(\mathcal{C}_{/\!\!/X}\right)_{\dagger} \times_{\mathcal{C}} \mathcal{C}_{/\!\!/Y} & \rightarrowtail & \mathsf{Fun}(\Delta^1,\mathcal{C})_{\dagger} \times_{\mathcal{C}} \mathsf{Fun}(\Delta^1,\mathcal{C}) \\
\downarrow & \lrcorner & \downarrow {\scriptstyle \mathsf{ev}_1 \times \mathsf{ev}_1} \\
* & \xrightarrow[(X,Y)]{} & \mathcal{C} \times \mathcal{C}
\end{array}$$

So that $\left(\mathcal{C}_{/(X\times Y)}\right)_{\ddagger}^{\simeq}$ can be identified with the maximal subgroupoid of the fiber above.

Now, $\mathsf{Span}(\mathcal{C},\mathcal{C}^{\dagger})(X,Y) = \{(X,Y)\} \times_{\mathsf{Span}(\mathcal{C})^{\simeq}\times \mathsf{Span}(\mathcal{C})^{\simeq}} \mathsf{Fun}(\Delta^1, \mathsf{Span}(\mathcal{C},\mathcal{C}^{\dagger}))^{\simeq}$. Again, we can use our canonical equivalence $\mathsf{Fun}(\Delta^1,\mathsf{Span}(\mathcal{C},\mathcal{C}^{\dagger}))^{\simeq} \simeq \mathsf{Fun}(\mathsf{Tw}(\Delta^1)^{\mathsf{op}},\mathcal{C})^{\simeq}_{\square,\dagger}$. As $Tw(\Delta^1)^{\mathsf{op}} = \Lambda^2_0 = \Delta^1 \coprod_{\{0\}} \Delta^1$, we have that

$$\mathsf{Fun}(\mathsf{Tw}(\Delta^1)^{\mathsf{op}},\mathcal{C})^{\simeq}_{\square,\dagger} = \mathsf{Fun}(\mathsf{Tw}(\Delta^1)^{\mathsf{op}},\mathcal{C})^{\simeq}_{\dagger} \cong \mathsf{Fun}(\Delta^1,\mathcal{C})^{\simeq}_{\dagger} \times_{\mathcal{C}^{\simeq}} \mathsf{Fun}(\Delta^1,\mathcal{C})^{\simeq}$$

where $\mathsf{Fun}(\Delta^1,\mathcal{C})^{\simeq}_{\dagger}$ is the full subcategory of $\mathsf{Fun}(\Delta^1,\mathcal{C})^{\simeq}$ spanned by arrows in $\mathcal{C}$ belonging to $\mathcal{C}^{\dagger}$. This fits into a diagram

$$\begin{array}{ccc}
\mathsf{Fun}(\Delta^1,\mathcal{C})^{\simeq}_{\dagger} \times_{\mathcal{C}^{\simeq}} \mathsf{Fun}(\Delta^1,\mathcal{C})^{\simeq} & \xrightarrow{\sim} & \mathsf{Fun}(\Delta^1,\mathsf{Span}(\mathcal{C},\mathcal{C}^{\dagger}))^{\simeq} \\
{\scriptstyle \mathsf{ev}_1\times\mathsf{ev}_1}\downarrow & & \downarrow{\scriptstyle \mathsf{ev}_0\times\mathsf{ev}_1} \\
\mathcal{C}^{\simeq} \times \mathcal{C}^{\simeq} & \rightarrowtail & \mathsf{Span}(\mathcal{C},\mathcal{C}^{\dagger})^{\simeq} \times \mathsf{Span}(\mathcal{C},\mathcal{C}^{\dagger}))^{\simeq}
\end{array}$$

As $\mathcal{C}^{\simeq} \times \mathcal{C}^{\simeq} \rightarrowtail \mathsf{Span}(\mathcal{C},\mathcal{C}^{\dagger})^{\simeq} \times \mathsf{Span}(\mathcal{C},\mathcal{C}^{\dagger})^{\simeq}$ is monic, we can identify (up to equivalence of infinity categories) the fibers

$$\left(\mathcal{C}_{/(X\times Y)}\right)_{\ddagger}^{\simeq} \simeq \{(X,Y)\} \times_{\mathcal{C}^{\simeq}\times\mathcal{C}^{\simeq}} \left(\mathsf{Fun}(\Delta^1,\mathcal{C})^{\simeq}_{\dagger} \times_{\mathcal{C}^{\simeq}} \mathsf{Fun}(\Delta^1,\mathcal{C})^{\simeq}\right)$$

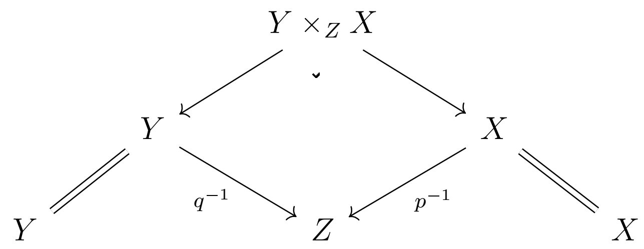
with

$$\{(X,Y)\} \times_{\mathsf{Span}(\mathcal{C},\mathcal{C}^\dagger)^\simeq \times \mathsf{Span}(\mathcal{C},\mathcal{C}^\dagger)^\simeq} \left(\mathsf{Fun}(\Delta^1,\mathcal{C})_\dagger^\simeq \times_{\mathcal{C}^\simeq} \mathsf{Fun}(\Delta^1,\mathcal{C})^\simeq\right) \simeq \mathsf{Span}(\mathcal{C},\mathcal{C}^\dagger)(X,Y)$$

(The last equivalence being a consequence of both sides being equivalent to the fiber $\{(X,Y)\} \times_{\mathsf{Span}(\mathcal{C},\mathcal{C}^\dagger)^\simeq \times \mathsf{Span}(\mathcal{C},\mathcal{C}^\dagger)^\simeq} \mathsf{Fun}(\Delta^1, \mathsf{Span}(\mathcal{C},\mathcal{C}^\dagger))^\simeq$) □

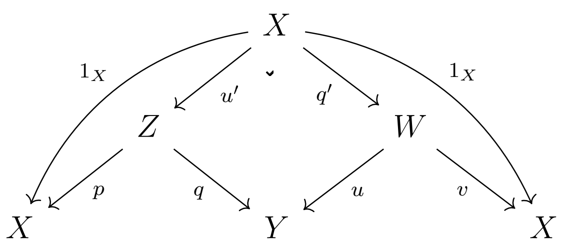
Proposition 1.29. *Consider a weak CoWaldhausen infinity category $(\mathcal{C},\mathcal{C}^\dagger)$. A 1-morphism of $\mathsf{Span}(\mathcal{C},\mathcal{C}^\dagger)$ seen as a span in $\mathcal{C}$ is invertible if and only if its legs are invertible.*

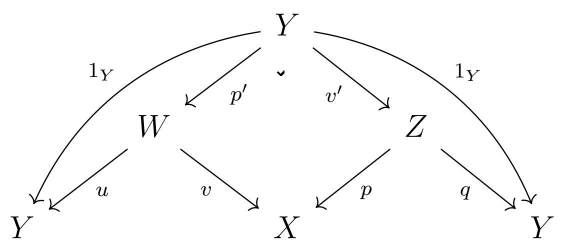
Proof. Let $X \xleftarrow{p} Z \xrightarrow{q} Y$ be a span (from X to Y) with invertible legs. One computes that if p^{-1} is an inverse to p and q^{-1} to q, then the composite



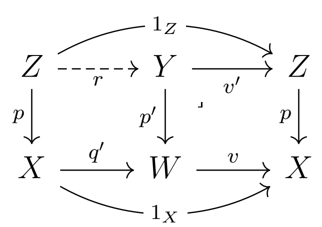
is an inverse in $\mathsf{Span}(\mathcal{C},\mathcal{C}^\dagger)$. Thus, every such span is an equivalence in $\mathsf{Span}(\mathcal{C},\mathcal{C}^\dagger)$. It suffices to show that every equivalence in $\mathsf{Span}(\mathcal{C},\mathcal{C}^\dagger)$ is represented by a span with both legs invertible.

To see this (we follow the proof of [Hau18, Lemma 8.2]), consider a span $X \xleftarrow{p} Z \xrightarrow{q} Y$ with inverse $Y \xleftarrow{u} W \xrightarrow{v} X$. Then there are witnesses to the composition being identity, i.e. 2-morphisms





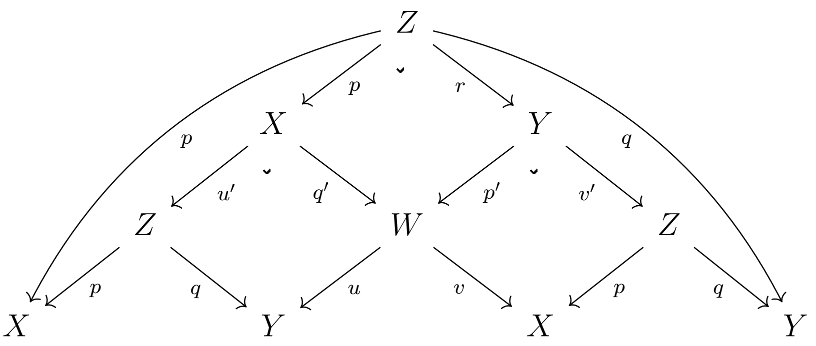
Consider the case for p, the other case of showing that q is invertible is dual. p has a section u', it suffices to show that p has a retract as well. Consider the diagram



The outer square is just the identity morphism of p, so Cartesian, hence the inner left square is as well. Thus there is a witness to the composition

$$(X \xleftarrow{p} Z \xrightarrow{q} Y) \circ (Y \xleftarrow{u} W \xrightarrow{v} X) \circ (X \xleftarrow{p} Z \xrightarrow{q} Y) = (X \xleftarrow{p} Z \xrightarrow{q} Y)$$

given by a diagram



in particular, we have a diagram of pullbacks

$$\begin{array}{ccccc} Z & \xleftarrow{u'} & X & \xleftarrow{p} & Z \\ {\scriptstyle q}\downarrow & & {\scriptstyle q'}\downarrow & & \downarrow{\scriptstyle r} \\ Y & \xleftarrow{u} & W & \xleftarrow{p'} & Y \end{array} \qquad (u \circ p' = 1_Y)$$

So that the witnessed composition $u' \circ p$ is an equivalence. Thus, p has a retract as well, so is an equivalence as desired. □

Theorem 1.30. *Let $\mathcal{C}^\dagger$ be a weak CoWaldhausen structure on an infinity category $\mathcal{C}$ with finite limits. Then, the Cartesian symmetric monoidal structure on $\mathcal{C}$ induces a natural symmetric monoidal structure on $\mathsf{Span}(\mathcal{C}, \mathcal{C}^\dagger)$ that acts via cartesian products "'levelwise" (however it is not the cartesian symmetric monoidal structure in general).*

Proof. Assertion (iv) of [Hau18, Theorem 1.2] is in fact a generalization of this statement to (∞, n)-categories of spans. □

2. The infinity categories $\mathcal{S}_n^m$

2.1. Recollection: n-finite spaces.

Notations 2.1. We denote by $\mathcal{S}$ the infinity category of spaces (or infinity groupoids) defined with respect to some implicit universe.

We also denote for $n \geqslant -1$ the n-sphere $\mathbb{S}^n \in \mathcal{S}$. We have defined $\mathbb{S}^{-1}$ to be the initial object $\varnothing$ in order to be consistent with the pushout diagrams

$$\begin{array}{ccc} \mathbb{S}^n & \longrightarrow & * \\ \downarrow & & \downarrow \\ * & \longrightarrow & \mathbb{S}^{n+1} \end{array}$$

in $\mathcal{S}$ for $n \geqslant -0$.

Remark 2.2. While the notion of an n-finite space was introduced much earlier, the sequence of results generally follows [Har20] here as well. However (particularly in the proofs) we have made at points as a matter of personal preference minor changes, and more extensive (albeit longer) explanations.

Definition 2.3. For $n \geqslant -2$, we call a space $X \in \mathcal{S}$ *n-truncated* if the diagonal map $X \to X^{\mathbb{S}^{n+1}}$ is an equivalence of spaces (weak homotopy equivalence).

A map $X \to Y$ of spaces is n-truncated when each (homotopy) fiber is n-truncated.

A space is *n-finite* if it is n-truncated and all its homotopy sets are finite. We call a space π-finite if it is n-finite for some n.

Observation 2.4. *A space is (-2)-truncated if and only if it is contractible. A space is (-1)-truncated precisely when either it is $\varnothing$ or contractible.*

Further, an application of the long exact sequence of homotopy groups shows that a space X is n-truncated for $n \geqslant 0$ if and only if for each basepoint $x \in X$ and $i > n$, the homotopy group $\pi_i(X, x)$ is zero.

Remark 2.5. With these definitions, Definitions 1.4 of a faithful functor can be restated as a functor $\mathcal{F}\colon \mathcal{C} \to \mathcal{D}$ such that for each pair of objects $X, X \in \mathcal{C}$, the action on Hom spaces, $\mathsf{Hom}_{\mathcal{C}}(X, Y) \to \mathsf{Hom}_{\mathcal{D}}(\mathcal{F}X, \mathcal{F}Y)$ is (-1)-truncated.

Notation 2.6. For $-2 \leqslant m \leqslant n$, let $\mathcal{S}_n \subseteq \mathcal{S}$ denote the full subcategory spanned by the n-finite spaces, and let $\mathcal{S}_{n,m} \subseteq \mathcal{S}$ denote the subcategory of n-finite spaces and m-truncated maps between them (in the sense of Construction 1.11 above).

Definition 2.7. For $n \geqslant -2$, denote by κ_n a set of weak homotopy equivalence classes of n-finite spaces. We will say that a category admits (resp. a functor preserves) n-finite or κ_n-(co)limits when it admits (resp. preserves) (co)limits of diagrams indexed by n-finite spaces, or equivalently elements of κ_n.

Observation 2.8. *Fiber products of n-finite spaces are n-finite. Consequently, the full subcategory $\mathcal{S}_n \subseteq \mathcal{S}$ has pullbacks (so finite limits, even), and these are computed as pullbacks in $\mathcal{S}$ itself.*

Proposition 2.9. *For $n \geqslant -2$, $\mathcal{S}_n$ has n-finite colimits, which are preserved and detected (or reflected) by the subcategory inclusion $\mathcal{S}_n \subseteq \mathcal{S}$.*

Proof. As $\mathcal{S}_n$ is a full subcategory of $\mathcal{S}$, the proposition will follow if we show that $\mathcal{S}_n$ is closed under n-finite colimits of n-finite spaces in $\mathcal{S}$.

Consider an arbitrary diagram $\mathcal{F}\colon X \to \mathcal{S}_n$, with X an n-finite space. Recall that we have a higher Grothendieck correspondence between functors out of an X valued in spaces and left fibrations over X (cf. [Cis19, §5.2, in particular Corollary 5.2.8]). The map $\mathcal{F}$, thought of as a functor $X \to \mathcal{S}$ classifies a Kan-fibration $p\colon E \twoheadrightarrow_{\mathsf{Kan}} X$ (a corollary of Joyal's coherence/lifting theorem is that a left fibration whose target is an infinity groupoid is a Kan-fibration, cf. [Cis19, Proposition 3.5.5]).

Lemma 2.10. *Consider a diagram $\mathcal{F}\colon X \to \mathcal{S}$, classifying a Kan-fibration $p\colon E \twoheadrightarrow_{\mathsf{Kan}} X$. The colimit of $\mathcal{F}$ is the homotopy type of the total space E.*

Proof. We can think of the colimit functor $\mathsf{Fun}(X, \mathcal{S}) \to \mathcal{S}$ as the left Kan extension $\pi_!\colon \mathsf{Fun}(X, \mathcal{S}) \to \mathsf{Fun}(*, \mathcal{S}) \simeq \mathcal{S}$ along the map to the terminal object $\pi\colon X \to *$.

In terms of left fibrations, this can be computed by forming a diagram (cf. [Cis19, dual of Proposition 6.1.14]):

$$\begin{array}{ccc} E & \xrightarrow{\mathsf{cofinal}} & \tilde{E} \\ {\scriptstyle p}\downarrow\!\!\downarrow & & \downarrow\!\!\downarrow \\ X & \xrightarrow[\pi]{} & * \end{array}$$

by factoring the composite $E \to X \to *$ into a cofinal/initial map and a left fibration. Then, $\pi_!\mathcal{F}$ is the homotopy type of $\tilde{E}$.

In our case, $X \twoheadrightarrow_{\mathsf{Kan}} *$ is a Kan-fibration, so no factorization is required and we directly compute $\pi_!\mathcal{F}$ as the homotopy type of $\tilde{E} := E$. $\square$

In light of the lemma, it suffices to show that the total space E is also n-finite. By assumption, the fiber of p at an object x has the homotopy type $\mathcal{F}_x \equiv \mathcal{F}(x)$, which is n-finite. As the total space of a Kan-fibration, with both total space and fiber n-finite, it is therefore the case that E is also n-finite (by the long exact sequence in homotopy). □

Corollary 2.11. *The colimit of a diagram $X \to \mathcal{S}_n$ for an n-finite space X is also given by the homotopy type of the Kan-fibration it classifies.*

Construction 2.12. For a space $X \in \mathcal{S}$, there is a canonical cocone (denoted $(i_x)_{x\in X}$, or when X is clear from context simply (i_x)) establishing X as the colimit of the constant X-indexed diagram in $\mathcal{S}$ with value the terminal object $*$. We will also write i_x for the cocone map associated to $x \in X$.

Intuitively, this is the inclusion of each point x, $i_x \colon * = \{x\} \rightarrowtail X$. More formally, the cocone $*_X \to X_X$ is the morphism in $\mathsf{Fun}(X, \mathcal{S})$ corresponding to the morphism of left-fibrations over X from 1_X (which classifies the constant functor at $*$) to $\pi_1 \colon X \times X \to X$ (which classifies the constant functor with value X) given by

$$\begin{array}{ccc} X & \xrightarrow{\delta_X} & X \times X \\ & \overset{1_X}{\searrow\!\!\searrow} & \big\downarrow{\scriptstyle \pi_1} \\ & & X \end{array}$$

(Observe that the fiber over $x \in X$ literally does pull out the map $* \to X$ with value x)

Proposition 2.13. *The cocone (i_x) of Construction 2.12 establishes X as the colimit of the constant X-indexed diagram with value $*$.*

Proof. Denoting by π again the map $X \to *$ and $\pi^* \colon \mathcal{S} \cong \mathsf{Fun}(*, \mathcal{S}) \to \mathsf{Fun}(X, \mathcal{S})$ the pullback along π, we can identify $*_X = \pi^*(*)$ and $Y_X = \pi^*(Y)$ for $Y \in \mathcal{S}$. If $Y = \pi_!(\pi^*(*))$ is the colimit of $*_X$, then the colimit cocone is just the component of the unit map $1 \to \pi^*\pi_!$ of the adjunction $\pi_! \dashv \pi^*$ at $\pi^*(*)$.

If, for a $\mathcal{F} \colon X \to \mathcal{S}$ and $f \colon X \to X'$, $f_!\mathcal{F}$ is computed by forming the diagram

$$\begin{array}{ccc} E & \xrightarrow{\text{cofinal}} & \tilde{E} \\ {\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle q} \\ X & \xrightarrow[f]{} & X' \end{array}$$

and taking the functor $X' \to \mathcal{S}$ classifying q, then the unit map $\mathcal{F} \to f^*f_!\mathcal{F}$ corresponds to the induced map $E \to X \times_{X'} \tilde{E}$ of left fibrations over X.

In our case, the diagram is

$$\begin{array}{ccc} X & \xrightarrow{1_X} & X \\ {\scriptstyle p}\big\downarrow & & \big\downarrow \\ X & \xrightarrow[\pi]{} & * \end{array}$$

and thus we get precisely the map claimed. □

Notation 2.14. Consider a map $p \colon E \to X$ of quasicategories (or simplicial sets, even). For an object $x \in X$, we define

$$\begin{array}{ccc} E_{/x} & \xrightarrow{p_{/x}} & X_{/x} \\ \big\downarrow & \lrcorner & \big\downarrow \\ E & \xrightarrow{p} & X \end{array}$$

Furthermore, for a functor $\mathcal{F}\colon E \to \mathcal{C}$, we denote by ${}^{\mathcal{F}}\!\!\diagup_{x}$ the composite

$$ {}^{E}\!\!\diagup_{x} \to E \xrightarrow{\mathcal{F}} \mathcal{C} $$

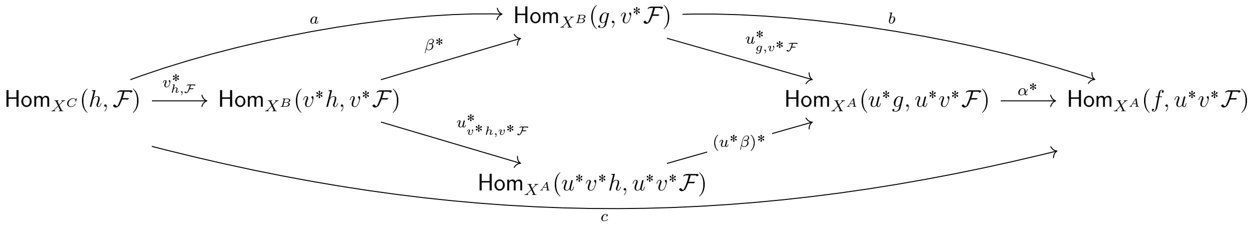
Lemma 2.15. *(Pasting) Consider a pair of composable functors $A \xrightarrow{u} B \xrightarrow{v} C$ of infinity categories, an infinity category X and functors $f\colon A \to X, g\colon B \to X, h\colon C \to X$. Consider also $\alpha\colon f \to u^*g$ in X^A, $\beta\colon g \to v^*h$ in X^B, and assume that α exhibits g as a left Kan-extension of f along u.*

Then, β exhibits h as a left Kan-extension of g along v if and only if the composite

$$ \gamma\colon f \xrightarrow{\alpha} u^*g \xrightarrow{u^*\beta} u^*v^*h \cong (vu)^*h $$

exhibits h as a left Kan-extension of f along vu.

Proof. We have by naturality the commutative diagram



Where α being a witness to a left Kan-extension translates to the map b being an equivalence. Now β is a witness to a left Kan-extension if and only if a is an equivalence. Similarly γ witnesses a left Kan-extension if and only if c is an equivalence. By the two out of three property, the lemma follows. □

Proposition 2.16. *Consider for $n \geqslant -2$ an infinity category $\mathcal{C}$ with n-finite colimits, and a functor $\mathcal{F}\colon \mathcal{S}_n \to \mathcal{C}$. Then, $\mathcal{F}$ preserves n-finite colimits if and only if for every object $X \in \mathcal{S}_n$, the image of the canonical cocone of Construction 2.12 under $\mathcal{F}$ is a cocone exhibiting $\mathcal{F}X$ as the colimit of the constant X indexed diagram with value $\mathcal{F}*$.*

Proof. We have seen that the canonical cocone of Construction 2.12 is a colimit cocone, so if $\mathcal{F}$ preserves n-finite colimits it preserves this one. For the converse, consider a diagram $u\colon X \to \mathcal{S}_n$ for Y an n-finite space. If u classifies a left fibration $p\colon E \twoheadrightarrow X$ with n-finite total space E, we have in Lemma 2.10 seen that there is a canonical colimit cocone $\alpha\colon u \to E_X$ in $\mathsf{Fun}(X, \mathcal{S}_n)$. We must show that $\mathcal{F}\alpha\colon \mathcal{F}u \to (\mathcal{F}E)_X$ is also a colimit cocone.

As $\mathcal{C}$ has n-finite colimits, it has sufficiently many colimits (cf. [Cis19, Proposition 6.4.9]) for the existence of the left Kan-extension functor

$$ \mathcal{C}^E \underset{p^*}{\overset{p_!}{\rightleftarrows}} \mathcal{C}^X \quad (p_! \dashv p^*) $$

Recall that in such a case, we can compute the left Kan-extension of any $\mathcal{F}\colon E \to \mathcal{C}$ by the identification

$$ (p_!\mathcal{F})_x = \mathsf{colim}\, {}^{\mathcal{F}}\!\!\diagup_{x} $$

Where we defined ${}^{\mathcal{F}}\!\!\diagup_{x}$ as in Notation 2.14.

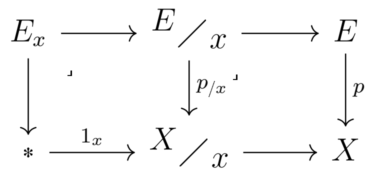
Lemma 2.17. *Consider a left fibration $p\colon E \twoheadrightarrow_{\mathsf{L}} X$ and an infinity category $\mathcal{C}$ with sufficiently many colimits for the existence of the left Kan-extension functor $p_!\colon \mathcal{C}^E \to \mathcal{C}^X$. Then we have for arbitrary $\mathcal{F}\colon X \to \mathcal{C}$ a canonical identification*

$$ (p_!\mathcal{F})_x = \mathsf{colim}\, \mathcal{F}_{|E_x} $$

for all $x \in X$.

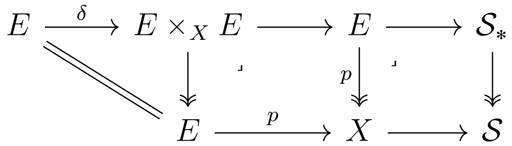
Remark 2.18. The same is true for every map between spaces, as long as we remember our convention of taking fibers in the infinity category of spaces, i.e. the "homotopy fiber".

Proof. We have the following diagram,



Now, recall (cf. [Cis19, Definition 4.4.1 and Proposition 4.4.11]) that as the left fibration p is proper and 1_x is final, the map $E_x \to {}^{E}\!/_{x}$ is final as well. Consequently the computation of a ${}^{E}\!/_{x}$ indexed limit can equivalently be done after pulling back to E_x ([Cis19, Proposition 6.4.5]). ☐

We claim that $\mathcal{F}u$ is the left Kan-extension of $(\mathcal{F}*)_E$ along p. In light of Lemma 2.17 on Kan-extensions, this is just the computation $(p_!\mathcal{F})_x \simeq \mathsf{colim}(\mathcal{F}*)_{E_x}$, which in light of the identification $E_x \simeq u(x)$ is $\mathsf{colim}(\mathcal{F}*)_{u(x)} \simeq \mathcal{F}(\mathsf{colim}\, *_{u(x)}) \simeq (\mathcal{F}u)_x$. Further, the witness $(\mathcal{F}*)_E \to p^*\mathcal{F}u = \mathcal{F}up$ is given by $\mathcal{F}$ being applied to the map $*_E \to up$ in $\mathsf{Fun}(E, \mathcal{S}_n)$ given by the morphism of fibrations



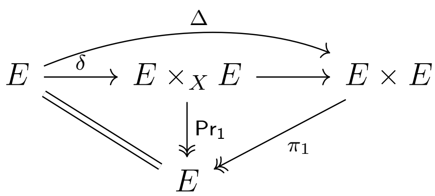
Further, the composite

$$(\mathcal{F}*)_E \to p^*(\mathcal{F}u) \xrightarrow{p^*\mathcal{F}\alpha} p^*q^*(\mathcal{F}E) = (\mathcal{F}E)_E$$

is obtained by applying $\mathcal{F}$ to the map

$$*_E \to p^*u \to E_E$$

which corresponds to the diagram of left fibrations,



and thus can be identified with (i_e) as well. By hypothesis the composite $\mathcal{F}(i_e)$ is a representing map of a left Kan-extension, and we have seen that $(\mathcal{F}*)_E \to p^*(\mathcal{F}u)$ is as well. Consequently, by the pasting Lemma 2.15, $\mathcal{F}\alpha$ represents $\mathcal{F}E$ as $q_!(\mathcal{F}u)$, that is the colimit of $\mathcal{F}u$. This concludes the proof. ☐

2.2. **The category of spans $\mathcal{S}_n^m$.**

Proposition 2.19. *For $-2 \leqslant m \leqslant n$, $(\mathcal{S}_n, \mathcal{S}_{n,m})$ is a weak CoWaldhausen infinity category .*

Proof. First, $\mathcal{S}_{n,m}$ is a wide subcategory of $\mathcal{S}_n$ as the equivalences in both are the equivalences in $\mathcal{S}_n$. Precisely, an application of the long exact sequence in homotopy shows that a 1-morphism in $\mathcal{S}_n$ is an equivalence if and only if all fibers are contractible. In other words, if and only if the 1-morphism lies in $\mathcal{S}_{n,-2}$. That is, we have inclusions of subcategories (on the nose)

$$\mathcal{S}_n^{\simeq} = \mathcal{S}_{n,-2} \subseteq \mathcal{S}_{n,m} \subseteq \mathcal{S}_n$$

so that taking the maximal subgroupoid inclusion yields an actual equality of subcategories $\mathcal{S}_n^{\simeq} \subseteq \mathcal{S}_{n,m}^{\simeq} \subseteq \mathcal{S}_n^{\simeq}$.

Now, as the property of being m-truncated is preserved by equivalences in the arrow category, it suffices to show that the pullback of m-truncated maps is m-truncated. The claim thus follows from the fact that we can think of fibers as forming pullbacks in spaces. As we can compose pullback squares, the fiber of the pullback is a fiber of the original map. Thus m-truncated maps pull back to m-truncated maps. The proposition follows. □

Definition 2.20. For $-2 \leqslant m \leqslant n$, let $\mathcal{S}_n^m := \mathsf{Span}(\mathcal{S}_n, \mathcal{S}_{n,m})$.

Remark 2.21. As $\mathcal{S}_n$ has finite limits, we may apply Theorem 1.30, establishing a symmetric monoidal structure on $\mathcal{S}_n^m$ for $-2 \leqslant m \leqslant n$, that acts by Cartesian product on objects and by level-wise product on 1-morphisms.

Observation 2.22. *By considering the long exact sequence in homotopy one also observes that any map between n-finite spaces is necessarily n-truncated. In other words $\mathcal{S}_{n,n} = \mathcal{S}_n$.*

Thus, we denote $\mathsf{Span}(\mathcal{S}_n) := \mathcal{S}_n^n = \mathsf{Span}(\mathcal{S}_n, \mathcal{S}_{n,n}) = \mathsf{Span}(\mathcal{S}_n, \mathcal{S}_n)$, the infinity category of spans of n-finite spaces (with no restriction on the wrong way maps).

We then have for $n \geqslant -2$ a filtration

$$\mathcal{S}_n \simeq \mathcal{S}_n^{-2} \subseteq \mathcal{S}_n^{-1} \subseteq \ldots \mathcal{S}_n^m \subseteq \ldots \mathcal{S}_n^n = \mathsf{Span}(\mathcal{S}_n)$$

Proposition 2.23. *For $-2 \leqslant m \leqslant n$, the subcategory inclusion $\mathcal{S}_n \subseteq \mathcal{S}_n^m$ preserves n-finite colimits.*

Proof. By Proposition 2.16 on n-finite colimit preserving functors out of $\mathcal{S}_n$, it suffices to show that the inclusion preserves the canonical colimit cocones $(i_x)\colon *_X \to X_X$ of Construction 2.12 for every $X \in \mathcal{S}_n$.

For this, we require the map in $\mathcal{S}^{\mathcal{S}_n^m}$

$$\mathsf{Hom}_{\mathcal{S}_n^m}(X, -) \xrightarrow{(-)_X} \mathsf{Hom}_{\mathcal{S}_n^{m\,X}}(X_X, (-)_X) \xrightarrow{(i_x)^*} \mathsf{Hom}_{\mathcal{S}_n^{m\,X}}(*_X, -)$$

to be an equivalence. It suffices to check that the component at an arbitrary $Y \in \mathcal{S}_n^m$ is invertible.

Lemma 2.24. *Consider $X \in \mathcal{S}$, and an infinity category $\mathcal{C}$ with X-shaped colimits. Then, for objects $a, b \in \mathcal{C}$, we have an identification*

$$\mathsf{Hom}_{\mathcal{S}}(X, \mathsf{Hom}_{\mathcal{C}}(a, b)) \simeq \mathsf{Hom}_{\mathcal{C}^X}(a_X, b_X)$$

Proof. As we can identify $\mathsf{Hom}_{\mathcal{S}}(*, -) \cong 1_{\mathcal{S}}$ and representable functors are continuous, we have

$$\mathsf{Hom}_{\mathcal{S}}(X, \mathsf{Hom}_{\mathcal{C}}(a, b)) = \mathsf{Hom}_{\mathcal{S}}(\mathsf{colim}_X *, \mathsf{Hom}_{\mathcal{C}}(a, b)) \simeq \mathsf{lim}_X \mathsf{Hom}_{\mathcal{S}}(*, \mathsf{Hom}_{\mathcal{C}}(a, b)) \simeq \mathsf{lim}_X \mathsf{Hom}_{\mathcal{C}}(a, b)$$

Now as the constant diagram a_X has a colimit in $\mathcal{C}$, this is in turn

$$\mathsf{lim}_X \mathsf{Hom}_{\mathcal{C}}(a, b) \simeq \mathsf{Hom}_{\mathcal{C}}(\mathsf{colim}_X a_X, b) \simeq \mathsf{Hom}_{\mathcal{C}^X}(a_X, b_X)$$

□

Thus, it suffices to prove that the composite

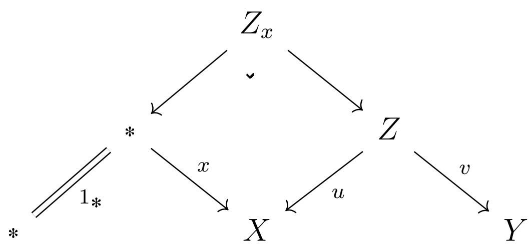
$$\mathsf{Hom}_{\mathcal{S}_n^m}(X, Y) \to \mathsf{Hom}_{\mathcal{S}_n^{m\,X}}(*_X, Y_X) \simeq \mathsf{Hom}_{\mathcal{S}}(X, \mathsf{Hom}_{\mathcal{S}_n^m}(*, Y))$$

is an equivalence. We have the identifications $\mathsf{Hom}_{\mathcal{S}_n^m}(X,Y) \simeq \left(\mathcal{S}_n \big/ X \times Y\right)^{\simeq}_{\{(u,v)\ |\ u:\mathcal{S}_{n,m}\}}$ of the mapping space onto the full subcategory of maps whose projection to X is m-truncated, and $\mathsf{Hom}_{\mathcal{S}}\left(X, \mathsf{Hom}_{\mathcal{S}_n^m}(*,Y)\right) \simeq \mathsf{Hom}_{\mathcal{S}}\left(X, (\mathcal{S}_n \big/ * \times Y)^{\simeq}_{\{(u,v)\ |\ u:\mathcal{S}_{n,m}\}}\right)$ similarly. We can also identify the term $\left(\mathcal{S}_n \big/ * \times Y\right)^{\simeq}_{\{(u,v)\ |\ u:\mathcal{S}_{n,m}\}}$ in the latter with $\left(\mathcal{S}_m \big/ Y\right)^{\simeq}$, as a map $Z \to *$ being m-truncated is equivalent to Z being m-finite.

Unraveling the identifications, one can understand the action of the composite

$$\left(\mathcal{S}_n \big/ X \times Y\right)^{\simeq}_{\{(u,v)\ |\ u:\mathcal{S}_{n,m}\}} \to \mathsf{Hom}_{\mathcal{S}}\left(X, \left(\mathcal{S}_m \big/ Y\right)^{\simeq}\right)$$

on objects as sending a $(u,v)\colon Z \to X \times Y$ to the X-indexed diagram of spans whose value at an object $x \in X$ is the composite span



This is in turn just taken to the X-indexed diagram $(Z_x \to Y)_{x \in X}$ in $\left(\mathcal{S}_m \big/ Y\right)^{\simeq}$ given by taking the right way maps. The fully coherent map can be described using the straightening construction, which is a functorial way of performing the assignment acting as $(Z \to X) \mapsto (Z_x)_{x\in X} \colon X \to \mathcal{S}$.

To be precise, we have an equivalence (cf. [Cis19, Corollary 6.5.9])

$$\mathsf{St}_X \colon \mathcal{S} \big/ X \xrightarrow{\sim} \mathsf{Fun}(X, \mathcal{S})$$

for every $X \in \mathcal{S}$. We construct a chain of equivalences

$$\mathcal{S} \big/ X \times Y \simeq \mathsf{Fun}(X \times Y, \mathcal{S}) \simeq \mathsf{Fun}(X, \mathsf{Fun}(Y, \mathcal{S})) \simeq \mathsf{Fun}\left(X, \mathcal{S} \big/ Y\right)$$

Now, $\mathsf{Hom}_{\mathcal{S}}(A,B) \simeq \mathsf{Fun}(A,B)^{\simeq} = \mathsf{Fun}(A,B)$ as all objects involved are infinity groupoids. Taking maximal subgroupoids and restricting to the $Z \to X \times Y$ whose projection to X is m-truncated (so Z is necessarily n-finite by a long exact sequence argument and the assumption $m \leqslant n$) we get an equivalence

$$\left(\mathcal{S}_n \big/ X \times Y\right)^{\simeq}_{\{(u,v)\ |\ u:\mathcal{S}_{n,m}\}} \to \mathsf{Hom}_{\mathcal{S}}\left(X, \left(\mathcal{S}_m \big/ Y\right)^{\simeq}\right)$$

which computes the aforementioned composite. This concludes the proof. □

Proposition 2.25. *Let $-2 \leqslant m \leqslant n$. Then the subcategory inclusion $\mathcal{S}_n \subseteq \mathcal{S}_n^m$ is wide.*

Proof. We must show that the $\mathcal{S}_n^{\simeq} \rightarrowtail \mathcal{S}_n^{m\,\simeq}$ is an equivalence. It acts as identity on objects so we need only show that it is fully-faithful.

Identifying $\mathcal{S}_n \simeq \mathcal{S}_n^{-2}$, by Proposition 1.22 on the action of such a functor on spans we see that for each $X, Y \in \mathcal{S}_n$ the map defines an equivalence of $\mathcal{S}_n$ onto the full subcategory of $\mathsf{Hom}_{\mathcal{S}_n^m}(X,Y)$ whose objects are the spans with (-2)-truncated wrong way maps (in other words, equivalences).

Lemma 2.26. *Consider an infinity category $\mathcal{C}$. For objects X, Y of $\mathcal{C}$, the mapping space $\mathsf{Hom}_{\mathcal{C}^{\simeq}}(X,Y)$ of $\mathcal{C}^{\simeq}$ can be identified with the full subcategory of $\mathsf{Hom}_{\mathcal{C}}(X,Y)$ spanned by the equivalences.*

Proof. For any particular X, Y, the map on mapping spaces induced by $\mathcal{C}^{\simeq} \rightarrowtail \mathcal{C}$ is up to equivalence the map

$$\mathsf{Map}^R_{\mathcal{C}^{\simeq}}(X,Y) \rightarrowtail \mathsf{Map}^R_{\mathcal{C}}(X,Y)$$

on right mapping spaces (recall from Notation 1.21 that the right mapping space $\mathsf{Map}^R_{\mathcal{C}}(X,Y)$ is the fiber of the projection $\mathcal{C}_{/Y} \to \mathcal{C}$ at X).

We have an inclusion $\mathcal{C}^{\simeq}_{/Y} \rightarrowtail \mathcal{C}_{/Y}$. The n-morphisms (n-simplices) of $\mathcal{C}_{/Y}$ can be described as $(n+1)$-morphisms of $\mathcal{C}$ terminating at Y. Under this inclusion, the n-morphisms that belong to $\mathcal{C}^{\simeq}_{/Y}$ are precisely those such that each consecutive map (restriction to $\Delta^{\{i,i+1\}} \cong \Delta^1$) is an equivalence. The fiber $\mathsf{Map}^R_{\mathcal{C}}(X,Y)$ (and similarly $\mathsf{Map}^R_{\mathcal{C}^{\simeq}}(X,Y)$) is computed by taking the $(n+1)$-simplices terminating in Y whose restriction to $\Delta^{\{0,1,\dots n\}}$ is the degenerate n-simplex at X. Consequently, the only determining property for a n-morphism of $\mathsf{Map}^R_{\mathcal{C}}(X,Y)$ to lie in $\mathsf{Map}^R_{\mathcal{C}^{\simeq}}(X,Y)$ is that of the restriction to $\Delta^{\{n,n+1\}}$ being an equivalence. In other words, $\mathsf{Map}^R_{\mathcal{C}^{\simeq}}(X,Y)$ is precisely the full subcategory of $\mathsf{Map}^R_{\mathcal{C}}(X,Y)$ spanned by the equivalences. □

The equivalence established immediately before the lemma therefore restricts to an equivalence of $\mathsf{Hom}_{\mathcal{S}_n^{\simeq}}(X,Y)$ onto the full subcategory of $\mathsf{Hom}_{\mathcal{S}_n^m}(X,Y)$ consisting of the spans whose legs are equivalences in $\mathcal{S}_n$. But we have seen in Proposition 1.29 that these are precisely the equivalences in $\mathcal{S}_n^m$. Thus, the map induces equivalences on mapping spaces, so is fully faithful as well. □

Corollary 2.27. *If X is a space, then every X-indexed diagram in $\mathcal{S}_n^m$ is equivalent to (the image of) an X-indexed diagram in $\mathcal{S}_n$.*

Observation 2.28. *A consequence of the fact that every diagram in $\mathcal{S}_n^m$ indexed by a space comes from one in $\mathcal{S}_n$ (Corollary 2.27), and the fact that the inclusion $\mathcal{S}_n \subseteq \mathcal{S}_n^m$ preserves n-finite colimits (Proposition 2.23) is that $\mathcal{S}_n^m$ admits n-finite colimits.*

Further, by Proposition 2.16 on colimit preserving functors out of $\mathcal{S}_n$, a functor $\mathcal{S}_n^m \to \mathcal{C}$ preserves n-finite colimits if and only if its restriction to $\mathcal{S}_n$ does.

Proposition 2.29. *The symmetric monoidal structure on $\mathcal{S}_n^m$ of Theorem 1.30 induced by the Cartesian product on $\mathcal{S}_n$ commutes with n-finite colimits in each variable.*

Proof. The Cartesian monoidal structure on $\mathcal{S}_n$ is the restriction of the one on $\mathcal{S}$, which commutes with colimits variable-wise. As $\mathcal{S}_n$ has n-finite colimits which are computed in $\mathcal{S}$ (Proposition 2.9), we see that the Cartesian monoidal structure on $\mathcal{S}_n$ preserves n-finite colimits.

We must check that, say for arbitrary n-finite X, the functor $X \otimes (-)\colon \mathcal{S}_n^m \to \mathcal{S}_n^m$ preserves n-finite colimits. The symmetric monoidal structure on $\mathcal{S}_n^m$ extends that on $\mathcal{S}_n$. In particular the restriction of $X \otimes (-)\colon \mathcal{S}_n^m \to \mathcal{S}_n^m$ to $\mathcal{S}_n$ (where X is taken as an object of $\mathcal{S}_n^m$) is equivalently the composite $X \otimes (-)\colon \mathcal{S}_n \to \mathcal{S}_n \rightarrowtail \mathcal{S}_n^m$ (with X now taken as an object of $\mathcal{S}_n$). As both factors are known by Proposition 2.23 to preserve n-finite colimits, this restriction commutes with them as well. We have just observed in Observation 2.28 that this is sufficient to conclude that the map $X \otimes (-)\colon \mathcal{S}_n^m \to \mathcal{S}_n^m$ commutes with n-finite colimits. □

Notation 2.30. For $n \geqslant -2$, we denote by Cat_{κ_n} the subcategory of Cat_∞ spanned by the infinity categories with n-finite colimits and functors preserving these colimits.

For infinity categories $\mathcal{C}$ and $\mathcal{D}$ with n-finite colimits, we denote by $\mathsf{Fun}_{\kappa_n}(\mathcal{C},\mathcal{D})$ the full subcategory of $\mathsf{Fun}(\mathcal{C},\mathcal{D})$ spanned by the functors preserving n-finite colimits.

Proposition 2.31. *(cf.* [Lur, Corollary 4.8.4.1]*) There is a symmetric monoidal structure* $\mathsf{Cat}_{\kappa_n}^{\otimes} \to \mathsf{Fin}_*$ *on* Cat_{κ_n} *such that for every two objects* $\mathcal{C}, \mathcal{D}$*, the tensor* $\mathcal{C} \otimes \mathcal{D}$ *is characterized by the existence of an n-cocontinuous map* $\mathcal{C} \times \mathcal{D} \to \mathcal{C} \otimes \mathcal{D}$ *such that for any other object* $\mathcal{E}$*, the restriction*

$$\mathsf{Fun}_{\kappa_n}(\mathcal{C} \otimes \mathcal{D}, \mathcal{E}) \to \mathsf{Fun}_{\kappa_n}(\mathcal{C} \times \mathcal{D}, \mathcal{E})$$

is fully faithful and induces an equivalence onto the full subcategory of $\mathsf{Fun}_{\kappa_n}(\mathcal{C} \times \mathcal{D}, \mathcal{E})$ *that are n-cocontinuous in each variable.*

Corollary 2.32. *The commutative algebra objects in* Cat_{κ_n} *can be identified with symmetric monoidal infinity categories admitting n-finite colimits such that the monoidal product is variable-wise n-cocontinuous.*

In particular, by Proposition 2.29 the symmetric monoidal infinity category $\mathcal{S}_n^m$ *is a commutative algebra object in* $\mathsf{Cat}_{\kappa_n}^{\otimes}$*.*

3. Higher Semiadditivity

3.1. **Idea.** To make sense of a notion of higher semiadditivity, we must first make precise the norm map that we wish to be an equivalence. Consider the first truly "higher" case, that is that of a diagram $\mathcal{F}\colon X \to \mathcal{C}$ with $\mathcal{C}$ 0-semiadditive and X a 1-finite space.

The data of a map $\mathsf{colim}\,\mathcal{F} \to \mathsf{lim}\,\mathcal{F}$ is intuitively that of a family of morphisms in $\mathcal{C}$, $\mathcal{F}x \to \mathcal{F}y$ that are functorial in the objects x, y of X.

To produce such a map, recall that when $\mathcal{C}$ is (0-)semiadditive, each mapping space $\mathsf{Hom}_{\mathcal{C}}(a, b)$ is an $\mathbb{E}_\infty$-monoid. In particular, when the mapping spaces $\mathsf{Hom}_X(x, y)$ of the indexing category are essentially discrete, we can define such a functorial family of maps $f_{x,y}\colon \mathcal{F}x \to \mathcal{F}y$ by simply "summing" the images of a choice of representatives of the path components of $\mathsf{Hom}_X(x, y)$. These correspond to a "Norm map"

$$\mathsf{Nm}_{\mathcal{F}}\colon \mathsf{colim}\,\mathcal{F} \to \mathsf{lim}\,\mathcal{F}$$

and we can try to define $\mathcal{C}$ to be 1-semiadditive when each norm map $\mathsf{Nm}_{\mathcal{F}}$ is invertible.

The summation operation is induced explicitly in terms of the Norm map for 0-semiadditivity[4]. For a finite family $f_1, f_2, \ldots, f_n\colon x \to y$, their summation is defined as

$$x \xrightarrow{\Delta} \mathsf{lim}\, x_X \to \mathsf{lim}\, y_X \simeq \mathsf{colim}\, x_X \xrightarrow{\nabla} y$$

where $\mathsf{lim}\, x_X \to \mathsf{lim}\, y_X$ is the map on colimits induced by $f_X\colon x_X \to y_X$ in $\mathsf{Fun}(X, \mathcal{C})$ (the "post-composition by f" functor), and the equivalence is the inverse of the norm map.

If $\mathcal{C}$ is 1-semiadditive, one can try to use the same definition to define an operation

$$\int_X\colon \mathsf{Fun}(X, \mathsf{Hom}_{\mathcal{C}}(x, y)) \to \mathsf{Hom}_{\mathcal{C}}(x, y)$$

which somehow gives a notion of summing or integrating families of morphisms from x to y in $\mathcal{C}$ indexed by 1-finite spaces.

Remark 3.1. This idea can be expanded upon to describe the notion of an m-commutative monoid. Intuitively, infinity categories where there is a notion of integrating families of morphisms with common source and target indexed by m-finite spaces.

One recovers $\mathbb{E}_\infty$-monoids as the 0-commutative monoid objects. One can show (cf. [Har20, Section 5.2]) that there is a sense in which that m-semiadditive infinity categories are those whose mapping spaces are m-commutative monoids.

[4]That is, the map "induced by the diagonal matrix"

Moving onward, one can try to repeat this for 2-semiadditivity. That is, for a 1-semiadditive $\mathcal{C}$, a 2-finite space X and a $\mathcal{F}\colon X \to \mathcal{C}$, one might attempt to define a functorial family of maps $f_{x,y}\colon \mathcal{F}x \to \mathcal{F}y$ by integrating the family of maps from $\mathcal{F}x$ to $\mathcal{F}y$ indexed by the 1-finite space $\mathsf{Hom}_X(x,y)$, that $\mathcal{F}$ defines. The rest of this section will be concerned with a formalization of precisely this approach.

Remark 3.2. The content of this section is primarily due to [HopLur13][Section 4].

3.2. **Ambidexterity.** Intuitively, for an n-finite space X and an infinity category $\mathcal{C}$ with n-finite colimits, one says that X is $\mathcal{C}$-ambidextrous if the colimit of any functor $X \to \mathcal{C}$ can be canonically identified with its limit. One then recovers the n-semiadditive infinity categories as those infinity categories $\mathcal{C}$ with n-finite colimits such that every n-finite space is $\mathcal{C}$-ambidextrous.

It will turn out to be convenient to see ambidexterity not as a property of spaces, but that of maps of spaces. That is, we would like to say that $f\colon X \to Y$ is $\mathcal{C}$-ambidextrous if the pullback functor $\mathsf{Fun}(Y,\mathcal{C}) \xrightarrow{f^*} \mathsf{Fun}(X,\mathcal{C})$ has a left adjoint $f_!$ which can be identified as the right adjoint as well. In other words, if the left and right Kan-extensions along f coincide. One of the main results of [HopLur13] is that the property of $\mathcal{C}$-ambidexterity is ultimately a condition on the (homotopy)-fibers of f, so in fact we can also identify $\mathcal{C}$ as being n-semiadditive when every morphism between n-finite spaces is $\mathcal{C}$-ambidextrous.

One minor inconvenience is that $\mathcal{C}$ may not have "large" enough colimits for the left Kan extension to be defined for all spaces (indeed, [HopLur13, Definition 4.4.2] only defines n-semiadditivity as a property of cocomplete infinity categories). Fortunately, as any infinity category with n-finite colimits has enough colimits for the left Kan extension of any $f\colon X \to Y$ between n-finite spaces to exist ("point-wise"), we can say that the functor of infinity categories $\mathcal{S}_n^{\mathsf{op}} \xrightarrow{\mathsf{Fun}(-,\mathcal{C})} \mathsf{Cat}_\infty$ takes any morphism to a right adjoint. This will suffice as [HopLur13] works with the notion of "Beck-Chevalley" fibrations, of which the Cartesian fibration corresponding to our functor $\mathcal{S}_n^{\mathsf{op}} \to \mathsf{Cat}_\infty$ is an example.

Notation 3.3. We will adhere to the standard notation of denoting the fiber of a map $X \to Y$ at an object $y\colon * \to Y$ as $X_y := \{y\} \times_Y X$. Further, when $\mathcal{C} \to \mathcal{X}$ is a biCartesian fibration[5] of quasicategories, we have a family of adjunctions parameterized by morphisms of $\mathcal{X}$. For a morphism $f\colon X \to Y$ of $\mathcal{X}$, we will denote this adjunction as

$$\mathcal{C}_X \underset{f^*}{\overset{f_!}{\rightleftarrows}} \mathcal{C}_Y \quad (f_! \dashv f^*)$$

Definition 3.4. If $\mathcal{C} \to \mathcal{X}$ is a Bicartesian fibration of quasicategories, we define for every commutative square σ in $\mathcal{X}$,

$$\begin{array}{ccc} A & \xrightarrow{u} & C \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ B & \xrightarrow[v]{} & D \end{array}$$

the *Beck-Chevalley* transformation $\mathsf{BC}[\sigma]\colon p_! u^* \to v^* q_!$ as the adjoint to the map

$$u^* \xrightarrow{u^*\eta_q} u^* q^* q_! \simeq p^* v^* q_!$$

(where $\eta_q\colon 1_{\mathcal{C}_C} \to q^* q_!$ is the unit of the adjunction induced by q)

[5] A map that is both a Cartesian and a coCartesian fibration.

Definition 3.5. If $\mathcal{X}$ is an infinity category with pullbacks, we say that a biCartesian fibration $\mathcal{C} \to \mathcal{X}$ is a *Beck-Chevalley fibration* when for every Cartesian square σ in $\mathcal{X}$, the Beck-Chevalley transformation $\mathsf{BC}[\sigma]$ of Definition 3.4 is invertible.

Example 3.6. The primary example of Beck-Chevalley fibrations that we will be concerned with is that of local systems in a category with n-finite colimits. That is, we will consider a category $\mathcal{C}$ with n-finite colimits, and the Cartesian fibration $\mathsf{LocSys}(\mathcal{C}) \to \mathcal{S}$ classifying the functor $\mathsf{Fun}(-, \mathcal{C})\colon \mathcal{S}^{\mathsf{op}} \to \mathsf{Cat}_\infty$. Its restriction/pullback to $\mathcal{S}_n$ is also co-Cartesian, classifying the functor that is the same on objects but is the left Kan extension on morphisms.

Definition 3.7. Given a biCartesian fibration $\mathcal{C} \to \mathcal{X}$, a morphism $X \xrightarrow{f} Y$ in $\mathcal{X}$, and a natural transformation $1_{\mathcal{C}_Y} \xrightarrow{\mu} f_! f^*$, we define for objects A, B of $\mathcal{C}_Y$ and a morphism $f^*A \xrightarrow{u} f^*B$ in $\mathcal{C}_X$ a $\int_f u d\mu\colon A \to B$ in $\mathcal{C}_Y$ by

$$A \xrightarrow{\mu_A} f_! f^* A \xrightarrow{f_! u} f_! f^* B \xrightarrow{\varepsilon_f} B$$

(where ε_f is the counit of the adjunction induced by f)

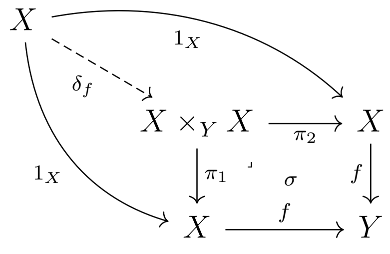
This induces a functor $d\mu\colon u \mapsto \int_f u d\mu$ given by the composition

$$\mathsf{Hom}_{\mathcal{C}_X}(f^*A, f^*B) \xrightarrow{(f_!)_{f^*A, f^*B}} \mathsf{Hom}_{\mathcal{C}_Y}(f_! f^*A, f_! f^*B) \xrightarrow{(\varepsilon_f)_*} \mathsf{Hom}_{\mathcal{C}_Y}(f_! f^*A, B) \xrightarrow{(\mu_A)^*} \mathsf{Hom}_{\mathcal{C}_Y}(A, B)$$

We will now introduce for every Beck-Chevalley fibration, a class of ambidextrous morphisms. We will do this by inductively constructing for every $n \geqslant -2$ a class of n-ambidextrous morphisms along with, for each n-ambidextrous f, a choice of natural translation $1_{\mathcal{C}_Y} \xrightarrow{\mu_f^n} f_! f^*$ exhibiting $f_!$ as a right adjoint to f^* (which is therefore well defined up to homotopy). For this it will be convenient to simultaneously define for each $n \geqslant -2$ a class of "weakly $(n+1)$-ambidextrous morphisms".

Construction 3.8. For a Beck-Chevalley fibration $\mathcal{C} \to \mathcal{X}$ such that the base $\mathcal{X}$ has pullbacks, call the class of equivalences in $\mathcal{X}$ the (-2)-ambidextrous morphisms. For every equivalence f, let μ_f^{-2} be a choice of inverse equivalence to the counit $f_! f^* \to 1_{\mathcal{C}_Y}$. For $n \geqslant -1$, given a notion of $(n-1)$-ambidextrous morphisms, we call a morphism $X \xrightarrow{f} Y$ weakly n-ambidextrous if the diagonal $X \xrightarrow{\delta_f} X \times_Y X$ is n-ambidextrous. Using this as a base case we will define for every $n \geqslant -2$ a notion of n-ambidextrous morphisms and μ^n's as above, and consequently a notion of weakly $(n+1)$-ambidextrous in terms of the diagonals.

Now, assuming that we have defined a notion of n-ambidextrous and the associated μ^n's, we will define for every weakly $(n+1)$-ambidextrous morphism f a natural transformation $\nu_f^{n+1}\colon f^* f_! \to 1_{\mathcal{C}_X}$. This is done by forming the diagram involving the pullback square σ:



and forming the composition

$$f^* f_! \xrightarrow{\mathsf{BC}[\sigma]^{-1}} \pi_{1!}\pi_2^* \xrightarrow{\pi_{1!}\mu_{\delta_f}^n \pi_2^*} \pi_{1!}\delta_{f!}\delta_f^*\pi_2^* \simeq 1_{\mathcal{C}_X} \circ 1_{\mathcal{C}_X} \simeq 1_{\mathcal{C}_X}$$

We will say that a morphism f of $\mathcal{X}$ is $(n+1)$-ambidextrous if every pullback f' of f along a morphism in $\mathcal{X}$ is weakly $(n+1)$-ambidextrous, and furthermore the associated $\nu_{f'}^{n+1}$ is the counit of an adjunction $f'^* \dashv f'_!$. In this case, we define μ_f^{n+1} to be a choice of unit for the adjunction $f^* \dashv f_!$ compatible with ν_f^{n+1}.

Remark 3.9. For an $(n+1)$-ambidextrous $f\colon X \to Y$, the component of ν_f^{n+1} at an object $A \in \mathcal{C}_X$ can be identified with $\int_{\delta_f} 1_{\mathcal{C}} d\mu_{\delta_f}^n \in \mathsf{Hom}_{\mathcal{C}_{X\times_Y X}}(\pi_2^* A, \pi_1^* A)$ under the equivalence of mapping spaces

$$\mathsf{Hom}_{\mathcal{C}_{X\times_Y X}}(\pi_2^* A, \pi_1^* A) \simeq \mathsf{Hom}_{\mathcal{C}_X}(\pi_{1!}\pi_2^* A, A) \simeq \mathsf{Hom}_{\mathcal{C}_X}(f^* f_! A, A)$$

We also note the following proposition.

Proposition 3.10. *(cf.* [HopLur13, Proposition 4.1.10]*) For a Beck-Chevalley fibration $\mathcal{C} \to \mathcal{X}$ such that the base $\mathcal{X}$ has pullbacks,*

(1) The classes of n-ambidextrous morphisms and weakly $(n+1)$-ambidextrous morphisms are closed under pullbacks.
(2) For $-1 \leqslant m \leqslant n$, if f is a weakly m-ambidextrous morphism, it is weakly n-ambidextrous as well and furthermore, ν_f^m and ν_f^n are homotopic.
(3) For $-2 \leqslant m \leqslant n$, if f is an m-ambidextrous morphism, it is n-ambidextrous as well and furthermore, μ_f^m and μ_f^n are homotopic.

Definition 3.11. For a Beck-Chevalley fibration $\mathcal{C} \to \mathcal{X}$ such that the base $\mathcal{X}$ has pullbacks (as in the above construction) we will say that a morphism in $\mathcal{X}$ is (weakly) ambidextrous if it is (weakly) n-ambidextrous for some n.

Remark 3.12. In light of the monotone nature of (weak) ambidexterity (cf. Proposition 3.10), we can coherently choose our μ_f^n's and ν_f^n's such that if f is m-ambidextrous and $n \geqslant m$, $\mu_f^m = \mu_f^n$ (and similarly for the ν_f^k's).

Thus, we may just speak of μ_f for f ambidextrous and ν_f for f weakly ambidextrous.

We record a lemma on the compatibility of the ν_f's and μ_f's with Beck-Chevalley transformations for future use.

Lemma 3.13. *(cf.* [HopLur13, Proposition 4.2.1]*) For an infinity category $\mathcal{X}$ with pullbacks, a Beck-Chevalley fibration $\mathcal{C} \to \mathcal{X}$, and a pullback square σ in $\mathcal{X}$*

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ {\scriptstyle p_X}\downarrow & \lrcorner \quad \sigma & \downarrow{\scriptstyle p_Y} \\ X & \xrightarrow[f]{} & Y \end{array}$$

(1) If f is weakly ambidextrous, so is f' and $\nu_{f'} p_X^*\colon f'^* f'_! p_X^* \to p_X^*$ is (homotopic to) the composition

$$f'^* f'_! f_X^* \xrightarrow{f'^* \mathsf{BC}[\sigma]} f'^* p_Y^* f_! \xrightarrow{\sim} p_X^* f^* f_! \xrightarrow{p_X^* \nu_f} p_X^*$$

(2) If f is ambidextrous, then so is f' and we also have that $p_Y^* \mu_f\colon p_Y^* \to p_Y^* f_! f^*$ is the composite

$$p_Y^* \xrightarrow{\mu_{f'} p_Y^*} f'_! f'^* p_Y^* \xrightarrow{\sim} f'_! p_X^* f^* \xrightarrow{\mathsf{BC}[\sigma] f^*} p_Y^* f_! f^*$$

3.3. **A criterion for ambidexterity.** We will now restrict to our case of interest, the Beck-Chevalley fibration $\mathsf{LocSys}(\mathcal{C}) \to \mathcal{S}_n$ for an infinity category $\mathcal{C}$ with n-finite colimits.

Here, $\mathsf{LocSys}(\mathcal{C})_X = \mathsf{Fun}(X, \mathcal{C})$ and the adjunction induced by a morphism $X \xrightarrow{f} Y$ is precisely the left Kan extension - pullback adjunction. Thus, our notion of ambidextrous morphisms does indeed realize morphisms for whom the left Kan extension is also a right Kan extension. For such a $\mathcal{C}$, we will call the (weakly) ambidextrous morphisms the (weakly) $\mathcal{C}$-ambidextrous morphisms.

Definition 3.14. For an infinity category $\mathcal{C}$ with n-finite colimits, call an n-finite space X (weakly) $\mathcal{C}$-ambidextrous if the map to the terminal object $X \to *$ in $\mathcal{S}_n$ is (weakly) $\mathcal{C}$-ambidextrous.

Remark 3.15. As the colimit (resp. limit) functor $\mathsf{Fun}(X, \mathcal{C}) \to \mathcal{C}$ can be identified with left (resp. right) Kan extension along the map from X to the terminal object, the colimit of a $\mathcal{C}$-valued functor out of a $\mathcal{C}$-ambidextrous space can be canonically identified as its limit.

The main result of [HopLur13] provides a characterization of $\mathcal{C}$-ambidextrous morphisms. We state a version of the result adapted for the case of an infinity category with n-finite colimits. While the proof is essentially the same, we present it just to be clear about the modifications to be made.

Proposition 3.16. *(cf.* [HopLur13, Proposition 4.3.5]*) Consider an infinity category $\mathcal{C}$ with n-finite colimits and $X \to Y$ a Kan fibration between n-finite spaces. Then f is $\mathcal{C}$-ambidextrous (resp. weakly ambidextrous) if and only if each fiber X_y is $\mathcal{C}$-ambidextrous (resp. weakly ambidextrous). In other words, for an infinity category $\mathcal{C}$ with n-finite colimits and a morphism $X \xrightarrow{f} Y$ in $\mathcal{S}_n$, f is $\mathcal{C}$-ambidextrous if and only if each (homotopy) fiber is $\mathcal{C}$-ambidextrous.*

Proof. The "only if" implication is a direct consequence of (weak) ambidexterity being stable under pullbacks (Proposition 3.10). For the converse, as every morphism between n-finite spaces is n-truncated, we may proceed by induction on $m := \min\{k \mid f \text{ is } k-\text{truncated}\}$.

The (-2)-truncated maps are the equivalences, and thus we get the base case $m = -2$ as equivalences are always (weakly) ambidextrous. Now for $m > -2$, assume that the claims hold for all m'-truncated morphisms with $m' < m$. First assume that each X_y is weakly ambidextrous, we will show the m-truncated f is weakly ambidextrous.

For this, note that the diagonal $\delta_f \colon X \to X \times_Y X$ is $(m-1)$-truncated, and further its fibers, say

$$\begin{array}{ccc} X_{(a,b)} & \longrightarrow & X \\ \big\downarrow & \lrcorner & \big\downarrow{\scriptstyle \delta_f} \\ * & \xrightarrow{(a,b)} & X \times_Y X \end{array}$$

for objects a, b of X with common image y in Y can be seen as the fiber

$$\begin{array}{ccc} X_{(a,b)} & \longrightarrow & X_y \\ \big\downarrow & \lrcorner & \big\downarrow{\scriptstyle \delta} \\ * & \xrightarrow{(a,b)} & X_y \times X_y \end{array}$$

so that $X_{(a,b)} \to *$ is the pullback of an ambidextrous morphism and hence ambidextrous. Thus by the $(m-1)$-truncated case applied to δ_f, f is weakly ambidextrous.

To complete the induction, we must also show that if the fibers X_y are in fact ambidextrous, then f is as well. Now, as the fibers of any pullback of f are fibers of f, it suffices to show that $\nu_f\colon f^*f_! \to 1_{\mathcal{C}^X}$ is a counit of an adjunction $f^* \dashv f_!$ (as we may replace f by any of its pullbacks).

What this means is that for any $\mathcal{F} \in \mathcal{C}_Y, \mathcal{G} \in \mathcal{C}_X$, the functorial map

$$\mathsf{Hom}_{\mathcal{C}^Y}(\mathcal{F}, f_!\mathcal{G}) \xrightarrow{f^*_{\mathcal{F},f_!\mathcal{G}}} \mathsf{Hom}_{\mathcal{C}^X}(f^*\mathcal{F}, f^*f_!\mathcal{G}) \xrightarrow{(\nu_f)_*} \mathsf{Hom}_{\mathcal{C}^X}(f^*\mathcal{F}, \mathcal{G})$$

is an equivalence. Note that this map is compatible with the formation of colimits in the $\mathcal{F} \in \mathcal{C}^Y$ variable.

Lemma 3.17. *Let $\mathcal{C}$ be an infinity category with n-finite colimits and $X \in \mathcal{S}_n$. Then, $\mathsf{Fun}(X, \mathcal{C})$ is generated under X-indexed (so in particular n-finite) colimits of functors of the form $\iota_!C$, where ι is the inclusion of an object $\{x\} \to X$ and $C \in \mathcal{C}$.*

Proof. (of the lemma) Consider the maps $X \xrightarrow{\delta} X \times X \xrightarrow{\pi_2} X$ composing to the identity. Then we have $\pi_{2!}\delta_! \simeq 1_{\mathcal{C}^X}$, and furthermore under $\mathsf{Fun}(X \times X, \mathcal{C}) \simeq \mathsf{Fun}(X, \mathsf{Fun}(X, \mathcal{C}))$ the functor $\pi_{2!}$ corresponds to $\mathsf{colim}\colon \mathsf{Fun}(X, \mathcal{C}^X) \to \mathcal{C}^X$.

Thus every object $F \in \mathcal{C}^X$ is a colimit of some $X \to \mathcal{C}^X$; in fact it is the colimit of the object $\mathcal{A}\colon X \to \mathcal{C}^X$ corresponding to $\delta_!\mathcal{F}$. The value of $\mathcal{A}$ at an object x can be computed as $(\iota_x \times 1_X)^*\delta_!\mathcal{F}\colon X \to \mathcal{C}$, where $\iota_x\colon \{x\} \to X$ is the inclusion of the point x. There is a Cartesian square with the Beck-Chevalley property,

$$\begin{array}{ccc} \{x\} & \xrightarrow{(1_{\{x\}}, \iota_x)} & \{x\} \times X \\ {\scriptstyle \iota_x}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle \iota_x \times 1_X} \\ X & \xrightarrow[\delta]{} & X \times X \end{array}$$

so we can compute

$$(\iota_x \times 1_X)^*\delta_!\mathcal{F} \simeq \iota_{x!}\iota_x^*\mathcal{F} = \iota_{x!}(\mathcal{F}_x)$$

Consequently, every such $\mathcal{F} = \mathsf{colim}_{x \in X}\, \iota_{x!}(\mathcal{F}_x)$ can be written as the colimit of an X-indexed diagram of objects of the form $\iota_!C$. ☐

Coming back to the proof of the main result, recall that we had noted right before the lemma that the map in question was compatible with colimits in the $\mathcal{C}^Y$ variable. The objects in question are precisely those generated by colimits as in Lemma 3.17. It thus suffices to show that the map is an equivalence when $\mathcal{F}$ is an object of the form $\iota_!C$, for $\iota\colon \{y\} \to Y$ the inclusion of an object and $C \in \mathcal{C}$.

Consider the following Cartesian square σ

$$\begin{array}{ccc} X_y & \xrightarrow{\iota_y} & X \\ \big\downarrow{\scriptstyle f_y} & \lrcorner \ \sigma & \big\downarrow{\scriptstyle f} \\ \{y\} & \xrightarrow[\iota]{} & Y \end{array}$$

There is a diagram

$$\begin{array}{ccccc} \mathsf{Hom}_{\mathcal{C}^Y}(\iota_!C, f_!\mathcal{G}) & \xrightarrow{f^*_{\iota_!C, f_!\mathcal{G}}} & \mathsf{Hom}_{\mathcal{C}^X}(f^*\iota_!C, f^*f_!\mathcal{G}) & \xrightarrow{(\nu_f)_*} & \mathsf{Hom}_{\mathcal{C}^X}(f^*\iota_!C, \mathcal{G}) \\ {\scriptstyle adj}\big\downarrow & & \big\downarrow & & \big\downarrow \\ \mathsf{Hom}_{\mathcal{C}}(C, \iota^*f_!\mathcal{G}) & \xrightarrow[(f_y^*)_{C, \iota^*f_!\mathcal{G}}]{} & \mathsf{Hom}_{\mathcal{C}^{X_y}}(f_y^*C, f_y^*\iota^*f_!\mathcal{G}) & \longrightarrow & \mathsf{Hom}_{\mathcal{C}^{X_y}}(f_y^*C, \iota_y^*\mathcal{G}) \\ \| & & \| & \nearrow{\scriptstyle (\nu_{f_y})_*} & \\ \mathsf{Hom}_{\mathcal{C}}(C, f_{y!}\iota_y^*\mathcal{G}) & \xrightarrow[(f_y^*)_{C, f_{y!}\iota_y^*\mathcal{G}}]{} & \mathsf{Hom}_{\mathcal{C}^{X_y}}(f_y^*C, f_y^*f_{y!}\iota_y^*\mathcal{G}) & & \end{array}$$

with the right square deriving from Lemma 3.13, which asserts the compatibility of the ν_f's with Beck-Chevalley transformations (the equalities with the third row are in light of the Beck-Chevalley isomorphisms). In particular, the right vertical map is invertible by the Beck-Chevalley property (and the left vertical one is also invertible as it is just an adjunction isomorphism).

Thus it suffices to show that the bottom path is invertible. But the composition is the adjunction morphism induced by ν_{f_y} for C and $\iota_y^*\mathcal{G}$. Invertibility thus follows from our assumption that each X_y is $\mathcal{C}$-ambidextrous, as this means that ν_{f_y} is indeed the counit of an adjunction $f_y^* \dashv f_{y!}$, completing the proof. □

3.4. **Semiadditivity.**

Definition 3.18. For $n \geqslant -2$, we define an infinity category $\mathcal{C}$ with n-finite colimits to be n-semiadditive when every n-finite space is $\mathcal{C}$-ambidextrous (with respect to the Beck-Chevalley fibration $\mathsf{LocSys}(\mathcal{C}) \to \mathcal{S}_n$).

Remark 3.19. In light of Proposition 3.16 (a corollary of the main result of Hopkins and Lurie), an infinity category $\mathcal{C}$ with n-finite colimits is n-semiadditive if and only if every morphism in $\mathcal{S}_n$ is $\mathcal{C}$-ambidextrous.

Definition 3.20. Consider an n-semiadditive infinity category $\mathcal{C}$, for every pair of objects A, B of $\mathcal{C}$ and n-finite space X we have Lemma 2.24, identifying

$$\mathsf{Hom}_{\mathcal{S}}(X, \mathsf{Hom}_{\mathcal{C}}(A, B)) \simeq \mathsf{Hom}_{\mathcal{C}^X}(A_X, B_X)$$

Given this and the integral map

$$d\mu_p = \int_p : \mathsf{Hom}_{\mathcal{C}^X}(p^*A, p^*B) \to \mathsf{Hom}_{\mathcal{C}}(A, B)$$

of Definition 3.7, we have a map

$$d\mu_X := \int_X : \mathsf{Hom}_{\mathcal{S}}(X, \mathsf{Hom}_{\mathcal{C}}(A, B)) \simeq \mathsf{Hom}_{\mathcal{C}^X}(A_X, B_X) \to \mathsf{Hom}_{\mathcal{C}}(A, B)$$

Examples 3.21. As the (-2)-finite spaces are precisely the contractible ones, every infinity category is (-2)-semiadditive, as promised.

As for (-1)-semiadditivity, the diagonal of $\varnothing$ is invertible so $\varnothing$ is always weakly ambidextrous. The pullback of $\varnothing \to *$ along $X \to *$ is simply the (essentially) unique $i\colon \varnothing \to X$. Further, the corresponding adjunction $i_! \dashv i^*$ is such that under the identification $\mathsf{Fun}(\varnothing, \mathcal{C}) \simeq *$ the right adjoint i^* is just the map to the terminal object, and the left adjoint $i_!\colon * \to \mathcal{C}^X$ picks out the initial object (constant functor at the initial object of $\mathcal{C}$).

The corresponding $\nu_i\colon i^*i_! \to 1_*$ is thus identifiable with the identity $i^*i_! = 1$, and this being a counit is just saying that the functorial map (in $\mathcal{F}\colon X \to \mathcal{C}$)

$$\mathsf{Hom}_{\mathcal{C}^X}(\mathcal{F}, \varnothing) = \mathsf{Hom}_{\mathcal{C}^X}(\mathcal{F}, i_!*) \to \mathsf{Hom}_*(i^*\mathcal{F}, i^*i!*) = *$$

is invertible, or in other words that the initial object of $\mathcal{C}^X$ is also final. Thus, we do indeed have that the (-1)-semiadditive infinity categories are the pointed ones.

Remark 3.22. More generally, a consequence of Proposition 3.16 (the main result of [HopLur13]) is that if $\mathcal{C}$ is n-semiadditive, every $(n+1)$-finite space is weakly $\mathcal{C}$-ambidextrous.

Remark 3.23. Now let $\mathcal{C}$ be pointed (i.e. (-1)-semiadditive), we compute for every pair of objects $A, B \in \mathcal{C}$ the associated $d\mu\colon * \simeq \mathsf{Hom}_{\mathcal{S}}(\varnothing, \mathsf{Hom}_{\mathcal{C}}(A, B)) \to \mathsf{Hom}_{\mathcal{C}}(A, B)$. This is precisely the zero map, as one of the objects in the composable sequence defining it is the zero object.

Computation 3.24. Consider a pointed infinity category $\mathcal{C}$, and a finite set X, thought of as a discrete space (equivalently, a 0-finite space). Denoting the map to the point $p\colon X \to *$, we wish to compute $\nu_X := \nu_p\colon p^*p_! \to 1_{\mathcal{C}}$. This can be done as the infinity category $\mathcal{C}$ is (-1)-semiadditive, as then the 0-finite X is weakly $\mathcal{C}$-ambidextrous.

For this we will first compute $\mu_\delta\colon 1_{\mathcal{C}} \to \delta_!\delta^*$, where $\delta\colon X \to X \times X$ is the diagonal of X (equivalently p). By our formula for computing left Kan extensions (Lemma 2.17), for a $\mathcal{F}\colon X \times X \to \mathcal{C}$, the value at a $(x, y) \in X \times X$ of $\delta_!\delta^*\mathcal{F}$ can be computed as $\mathsf{colim}_{X_{(x,y)}} \delta^*\mathcal{F}$. But we have

$$X_{(x,y)} \simeq \begin{cases} * & x = y \\ \varnothing & x \neq y \end{cases}$$

so that

$$(\delta_!\delta^*\mathcal{F})_{(x,y)} = \begin{cases} \mathcal{F}_x & x = y \\ 0 & x \neq y \end{cases}$$

with

$$(\mu_\delta\mathcal{F}\colon \mathcal{F} \to \delta_!\delta^*\mathcal{F}) \simeq \begin{cases} 1_{\mathcal{F}_x} & x = y \\ 0 & x \neq y \end{cases}$$

Given μ_δ, the transformation ν_X can be written as the composite

$$p^*p_! \simeq \pi_{1!}\pi_2^* \xrightarrow{\pi_{1!}\mu_\delta\pi_2^*} \pi_{1!}\delta_!\delta^*\pi_2^* \simeq 1_{\mathcal{C}}$$

with the first equivalence being induced by the Beck-Chevalley transformation associated to the Cartesian square

$$\begin{array}{ccc} X \times X & \xrightarrow{\pi_2} & X \\ {\scriptstyle\pi_1}\downarrow & \lrcorner & \downarrow{\scriptstyle p} \\ X & \xrightarrow{p} & * \end{array}$$

Now, we aim to compute for a functor $\mathcal{F}\colon X \to \mathcal{C}$ and an object $x \in X$, the map $(\nu_X\mathcal{F})_x : (p^*p_!\mathcal{F})_x = \mathsf{colim}_X \mathcal{F} \to \mathcal{F}_x$. For this we observe that for $\mathcal{G}\colon X \times X \to \mathcal{C}$, $\pi_{1!}\mathcal{G}$ can be computed as

$$(\pi_{1!}\mathcal{G})_x = \mathsf{colim}_{\{x\}\times X} \left(\mathcal{G}|_{\{x\}\times X}\right)$$

and we need simply apply that to (for y an arbitrary object of X)

$$(\pi_2^*\mathcal{F})_{(x,y)} \xrightarrow{(\mu_\delta\pi_2^*\mathcal{F})_{(x,y)}} (\delta_!\delta^*\pi_2^*\mathcal{F})_{(x,y)} \simeq (\delta_!\mathcal{F})_{(x,y)} = \begin{cases} 1_{\mathcal{F}_y} & x = y \\ 0 & x \neq y \end{cases}$$

Thus,

$$(\nu_X\mathcal{F})_x : \mathsf{colim}_X \mathcal{F} = (p^*p_!\mathcal{F})_x \simeq (\pi_{1!}\pi_2^*\mathcal{F})_x \xrightarrow{(\pi_{1!}\mu_\delta\pi_2^*\mathcal{F})_x} (\pi_{1!}\delta_!\mathcal{F})_x \simeq \mathcal{F}_x$$

is precisely the map whose restriction to the component corresponding to the object $x \in X$ is the identity, and whose restriction to every other component is the zero map.

The adjunction map

$$\mathsf{Hom}_{\mathcal{C}}\left(A, \coprod_{x\in X} \mathcal{F}_x\right) \xrightarrow{p^*_{A,\coprod_{x\in X}\mathcal{F}_x}} \mathsf{Hom}_{\mathcal{C}^X}\left(A_X, \left(\coprod_{x\in X} \mathcal{F}_x\right)_X\right) \xrightarrow{(\nu_X\mathcal{F})_*} \mathsf{Hom}_{\mathcal{C}^X}(A_X, \mathcal{F})$$

being an equivalence for arbitrary $\mathcal{F}$[6], is thus equivalent to $\nu_X\mathcal{F}$ exhibiting $\coprod_{x\in X} \mathcal{F}_x$ as the product/limit of $\mathcal{F}$.

[6]In other words, X being $\mathcal{C}$-ambidextrous.

Observation 3.25. *The last conclusion of Computation 3.24 can be restated as, X being $\mathcal{C}$-ambidextrous if and only if for arbitrary $\mathcal{F}\colon X \to \mathcal{C}$, $\mathcal{F}$ having a product $\prod_X \mathcal{F}$ in $\mathcal{C}$ and the canonical map $\coprod_X \mathcal{F} \to \prod_X \mathcal{F}$ given by the diagonal matrix being invertible, as promised.*

Proposition 3.26. *(cf.* [HopLur13, Proposition 4.4.9]*) Let $\mathcal{C}$ be a pointed infinity category with finite coproducts (equivalently, 0-finite colimits). Then $\mathcal{C}$ is 0-semiadditive if and only if for every pair of objects A, B, there is a product $A \times B$ in $\mathcal{C}$ and the canonical map $A \coprod B \to A \times B$ given by the diagonal matrix is invertible*[7].

Proof. The statement about products is just the two point space being ambidextrous, so it follows immediately from 0-semiadditivity. For the converse, assume that every such diagonal matrix map is invertible. The 0-finite spaces can be equivalently identified with finite discrete spaces or finite sets. In other words, it suffices to show that for every finite set X, the map $p\colon X \to *$ is $\mathcal{C}$-ambidextrous.

We know that pointed infinity categories are (-1)-semiadditive, so the empty set and the one element set are $\mathcal{C}$-ambidextrous. It in fact is sufficient to show that the two element set $X = \{0, 1\}$ is $\mathcal{C}$-ambidextrous, as we can then show it for any other finite set Y by induction on its cardinality (cf. the proof of [HopLur13, Proposition 4.4.9]).

To be precise, for Y with more than two points, pick one of them, say y and define $Y \to X$ sending y to 0 and all other points to 1. Then, this map has fibers given by sets of strictly smaller cardinality, which we can assume to be $\mathcal{C}$-ambidextrous by induction. Thus, by Proposition 3.16 (the main result of [HopLur13]), the map $Y \to X$ is also $\mathcal{C}$-ambidextrous. Consequently, if we can show that X is $\mathcal{C}$-ambidextrous, we may conclude that Y is as well. But the hypothesis lets us conclude that the two point space is indeed ambidextrous in light of the characterization of 0-ambidextrous sets of Observation 3.25. □

Remark 3.27. Thus, 0-semiadditive infinity categories are indeed the classical semiadditive infinity categories as promised.

Furthermore, the induced transformation

$$\int_X : \mathsf{Hom}_{\mathcal{S}}(X, \mathsf{Hom}_{\mathcal{C}}(A, B)) \to \mathsf{Hom}_{\mathcal{C}}(A, B)$$

can be seen to be just addition (in the sense of the $\mathbb{E}_\infty$-monoid operation on $\mathsf{Hom}_{\mathcal{C}}(A, B)$), so our so called "summation/integration" operation does indeed deserve the name in the 0-semiadditive case (when X is contractible, this acts as identity and when X is empty, it picks out the zero map). For general n, the $\int$ operation continues to act in such a way, "integrating" families of morphisms $A \to B$ in an n-semiadditive infinity category indexed by an n-finite space.

To be precise, one can (cf. [Har20, Section 5.2]) introduce a notion of n-commutative monoids, which generalize $\mathbb{E}_\infty$-monoids and are informally objects in which one can "integrate" over n-finite spaces in a coherent way. One shows that the Hom-spaces of n-semiadditive infinity categories are in fact n-commutative monoids, and the operation itself can be computed precisely by the $\int$-construction so defined.

Observation 3.28. *The opposite of an n-semiadditive infinity category is also n-semiadditive.*

Examples 3.29. We conclude by listing a few notable examples of semiadditivity.

- Stable infinity categories and nerves of semiadditive 1-categories are both 0-semiadditive.

[7]In other words, the canonical maps $A \coprod B \to A$ and $A \coprod B \to B$ given by the identity in one leg and zero in the other exhibit $A \coprod B$ as the product of A and B.

- (cf. [Har20, Corollary 3.20]) For all integers $-2 \leqslant m \leqslant n$ the infinity category $\mathcal{S}_n^m$ is m-semiadditive (also presented later on in Example 4.18).
- (cf. [Har20, Proposition 5.26]) For any $n \geqslant -2$, the category Cat_{κ_n} is n-semiadditive (cf. Proposition 5.12 for a slightly different proof than that in [Har20]).
- The category of $\mathbb{E}_\infty$-monoid objects in any infinity category with finite products is 0-semiadditive. (cf. [Har20, Section 5.2], also with a generalization)
- (cf. [HopLur13, Theorem 5.2.1]) For any $n \in \mathbb{N}$, the category of $K(n)$-local spectra is m-semiadditive for any m (where $K(n)$ denotes the Morava K-theory spectrum at height n).
- (cf. [CarSchYan18, Theorem 5.3.9]) For any $n \in \mathbb{N}$ and integer prime p, the infinity category of $T(n)$-local spectra is m-semiadditive for every m (where $T(n)$ is the telescope of a finite p-local type n spectrum).

4. A Criterion for Semiadditivity

We now proceed to establish an alternative definition for n-semiadditivity, following [Har20]. To be precise, we wish to establish the following theorem.

Theorem 4.1. *(cf.* [Har20, Corollary 3.19 and the discussion at the start of the first paragraph of section 5.1]*) Let $\mathcal{C}$ be an infinity category with n-finite colimits. Then $\mathcal{C}$ is n-semiadditive if and only if it admits an action of $\mathcal{S}_n^n$ compatible with n-finite colimits (that is, an action such that the action functor $\mathcal{S}_n^n \times \mathcal{C} \to \mathcal{C}$ commutes with n-finite colimits in each variable*[8]*).*

Remark 4.2. In fact, when an infinity category with n-finite colimits is n-semiadditive, the $\mathcal{S}_n^n$-action so described is essentially unique (cf. [Har20, Corollary 5.3], which is Corollary 5.10 below).

We will prove the equivalence of definitions in two steps. In this section we will prove that an infinity category with n-finite colimits and such an action of $\mathcal{S}_n^n$ is indeed n-semiadditive. This will be done by identifying the ν_f's and μ_δ's of Hopkins and Lurie in [HopLur13] with alternative natural transformations defined in terms of the $\mathcal{S}_n^n$-action. We will then show that the ν's defined in terms of the $\mathcal{S}_n^n$-action do indeed serve as counits for our desired adjunctions (cf. [Har20, Section 3]).

The converse direction is an immediate consequence of the universal property of $\mathcal{S}_n^n$ introduced by Harpaz in [Har20, Section 4] (where in fact, a universal property of $\mathcal{S}_n^m$ for general $-2 \leqslant m \leqslant n$ is introduced).

We will show that an infinity category with n-finite colimits admitting an action of $\mathcal{S}_n^n$ compatible with these colimits is n-semiadditive by induction on $n \geqslant -2$. The $n = -2$ case is immediate as all infinity categories are (-2)-semiadditive.

So it remains to establish the inductive step for $n \geqslant -1$. For this, it is standard to consider an $(n-1)$-semiadditive infinity category with n-finite colimits and such an action of $\mathcal{S}_n^n$, and then prove that it must be n-semiadditive. We will in fact establish, given a slightly weaker hypothesis, a more general (yet verbose) condition for such an infinity category to be n-semiadditive. Therefore, it is more convenient to assume instead the "standing hypothesis".

4.1. The ***Standing Hypothesis***.

Hypothesis 4.3. We consider an $(n-1)$-semiadditive infinity category $\mathcal{C}$ with n-finite colimits, equipped with an action of $\mathcal{S}_n^{n-1}$ compatible with n-finite colimits.

[8] Alternatively, that $\mathcal{C}$ is an $\mathcal{S}_n^n$-module in Cat_{κ_n}

We will say that such an infinity category $\mathcal{C}$ *satisfies the standing hypothesis* (with respect to this n).

Remark 4.4. Note that an action of $\mathcal{S}_n^n$ on an infinity category with n-finite colimits that is compatible with n-finite colimits restricts (by Observation 2.28) to an action of $\mathcal{S}_n^{n-1}$ that is also compatible with n-finite colimits. In particular, the *standing hypothesis* of Hypothesis 4.3 is indeed weaker than the inductive one.

As a consequence of Proposition 3.16 (the main result of Hopkins and Lurie), when $\mathcal{C}$ satisfies the standing hypothesis, every $X \in \mathcal{S}_n$ is weakly $\mathcal{C}$-ambidextrous. In particular, the inductive construction produces a ν_X for every X.

Similarly, every $(n-1)$-truncated map f in $\mathcal{S}_n$ is $\mathcal{C}$-ambidextrous, so we have an associated μ_f as well.

Notation 4.5. For the sake of convenience, we will denote the ν's and μ's introduced by this construction with indices, for instance ν^k or μ^k to indicate that these derive from an inductive construction, and to distinguish them from the morphisms ν and μ we will later define in terms of the action of spans.

Notation 4.6. Given an infinity category $\mathcal{C}$ satisfying the standing hypothesis (Hypothesis 4.3), given an n-finite space X we will denote by $[X] : \mathcal{C} \to \mathcal{C}$ the n-finite colimit preserving action of X.

Observation 4.7. *Recall that for any n-finite space X, we have as in Construction 2.12 a canonical cocone exhibiting X as the colimit of the X-indexed constant diagram at the point. Denoting by $p\colon X \to *$ the map to the point, this statement is an equivalence $X \simeq \tilde{p}_! \tilde{p}^* *$, in terms of the adjunction*

$$\mathcal{S}_n^X \underset{\tilde{p}^*}{\overset{\tilde{p}_!}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathcal{S}$$

induced by p. The action functor being n-cocontinuous in the $\mathcal{S}_n^{n-1}$-variable means that there is an induced identification

$$[X] \simeq [\tilde{p}_! *_X] \simeq p_! [*_X] = p_! p^* [*] \simeq p_! p^*$$

in terms of the adjunction

$$\mathcal{C}^X \underset{p^*}{\overset{p_!}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathcal{C}$$

also induced by p (the last equivalence a consequence of $[*] \simeq 1_{\mathcal{C}}$).

Remark 4.8. In light of the Observation 4.7 on the action of a space, our notation can be identified with that of [HopLur13, Notation 5.1.9]

4.2. **Trace Forms.** The criterion we ultimately establish will build upon a formulation in terms of "Trace Forms", as in [HopLur13, Section 5.1].

Notation 4.9. (cf. [HopLur13, Notation 5.1.7]) For a Beck-Chevalley fibration (recall Definition 3.5) $\mathcal{X} \xrightarrow{\pi} \mathcal{C}$ and a $X \xrightarrow{f} Y$ in $\mathcal{C}$, let $[X/Y]$ denote the composite

$$\mathcal{C}_Y \xrightarrow{f^*} \mathcal{C}_X \xrightarrow{f_!} \mathcal{C}_Y$$

Definition 4.10. (cf. [HopLur13, Notation 5.1.7]) For a Beck-Chevalley fibration $\mathcal{X} \xrightarrow{\pi} \mathcal{C}$ and a weakly ambidextrous $X \xrightarrow{f} Y$ in $\mathcal{C}$, define a map $\mathsf{TrFm}_f\colon [X/Y] \circ [X/Y] \to 1_{\mathcal{C}_Y}$, called the *Trace Form*, as the composite

$$[X/Y] \circ [X/Y] = f_! f^* f_! f^* \xrightarrow{f_! \nu_f^k f^*} f_! f^* \xrightarrow{\varepsilon_f} 1_{\mathcal{C}_Y}$$

where ε_f is the counit of the adjunction $f_! \dashv f^*$.

For a weakly ambidextrous object X of $\mathcal{C}$, we will define $\mathsf{TrFm}_X := \mathsf{TrFm}_p$, where $X \xrightarrow{p} *$ is the map to the terminal object.

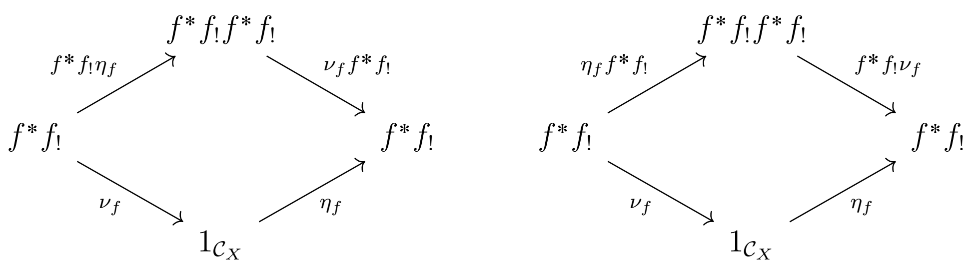
Proposition 4.11. *(cf.* [HopLur13, Proposition 5.1.8]*) For $\mathcal{X} \xrightarrow{\pi} \mathcal{C}$ a Beck-Chevalley fibration and a weakly ambidextrous $X \xrightarrow{f} Y$ in $\mathcal{C}$, the following are equivalent:*

- *ν_f^k is a counit of an adjunction $f^* \dashv f_!$, that is, f is ambidextrous.*
- *The Trace Form TrFm_f exhibits $[X/Y]$ as self dual in $\mathsf{Fun}(\mathcal{C}_Y, \mathcal{C}_Y)$.*

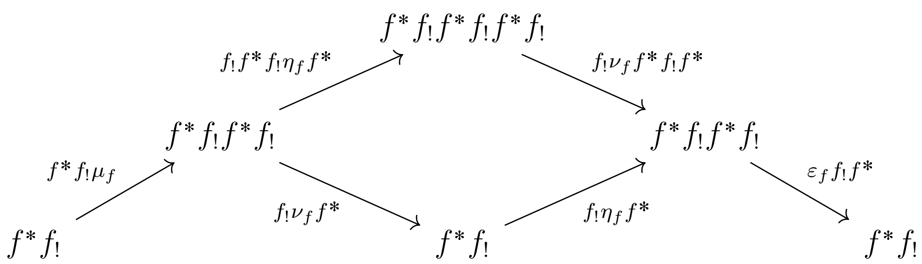
Proof. Assume first that f is ambidextrous. Then we have a unit $\mu_f := \mu_f^k$ compatible with $\nu_f := \nu_f^k$ in terms of which we can define a map $\mathsf{coev}\colon 1_{\mathcal{C}_Y} \to [X/Y]^2$ by

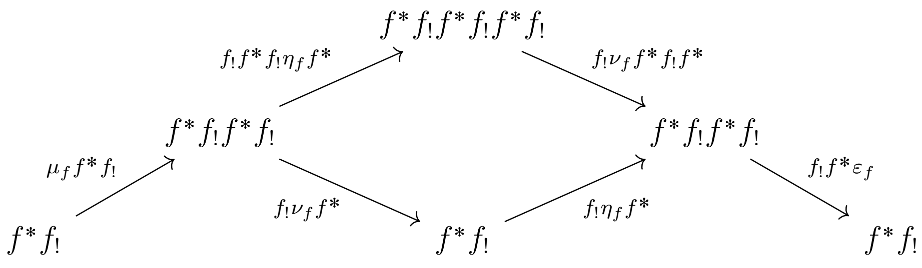
$$\mathsf{coev}\colon 1_{\mathcal{C}_Y} \xrightarrow{\mu_f^k} f_! f^* \xrightarrow{f_!\eta_f f^*} f_! f^* f_! f^* = [X/Y] \circ [X/Y]$$

where η_f is a unit for $f_! \dashv f^*$ compatible with ε_f. We claim that $\mathsf{TrFm}_f, \mathsf{coev}$ are an evaluation-coevaluation pair. To see this, we observe that we have commuting diagrams



which fit into diagrams





which precisely compute the composites which we we want to identify with the identity. In both cases, the bottom four maps are such that the first two and last two are both derived from triangle maps corresponding to the adjunctions (μ_f, ν_f) and (η_f, ε_f). Thus the identity is ultimately a composite of both, as desired.

For the converse, assume that TrFm_f is a self-duality evaluation map and let coev be a coevaluation compatible with it. We must show that $\nu_f := \nu_f^k$ is a counit, that is, that there is a unit map μ_f such that μ_f and ν_f satisfy the triangle identities.

It in fact suffices to find two morphisms $\mu_1, \mu_2 \colon 1_{\mathcal{C}_Y} \to f_! f^*$ such that each satisfies one of the triangle identities, that is, such that

$$\begin{array}{ccc} & f_! f^* f_! & \\ {\scriptstyle \mu_1 f_!} \nearrow & & \searrow {\scriptstyle f_! \nu_f} \\ f_! & = & f_! \end{array} \qquad \begin{array}{ccc} & f^* f_! f^* & \\ {\scriptstyle f^* \mu_2} \nearrow & & \searrow {\scriptstyle \nu_f f^*} \\ f^* & = & f^* \end{array}$$

This is because given such μ_1, μ_2 one sees that $\mu \colon 1_{\mathcal{C}_Y} \to f_! f^*$ defined as

$$\mu \colon 1_{\mathcal{C}} \xrightarrow{\mu_1 \mu_2} f_! f^* f_! f^* \xrightarrow{f_! \nu_f f^*} f_! f^*$$

can be seen to satisfy both the triangle identities with respect to ${\nu_f}$[9].

It remains to construct such a μ_1 and μ_2. We must show that $(f_! \nu_f) \circ (\mu_1 f_!) \simeq 1_{f_!}$ and $(\nu_f f^*) \circ (f^* \mu_2) \simeq 1_{f^*}$. For this it suffices to show that the respective adjoints under $f_! \dashv f^*$ are homotopic[10].

For instance, in the case of μ_2, we must show that $\varepsilon_f \circ (f_! \nu_f f^*) \circ (f_! f^* \mu_2)$ is homotopic to ε_f. This composite is simply $\mathsf{TrFm}_f \circ ([X/Y]\, \mu_2)$. In light of the evaluation-coevaluation identities, one may consider defining $\mu_2 := 1_{\mathcal{C}_Y} \xrightarrow{\mathsf{coev}} [X/Y]^2 \xrightarrow{[X/Y]\varepsilon_f} [X/Y]$. This does indeed work, in light of the diagram

$$\begin{array}{ccccc} [X/Y] & \xrightarrow{[X/Y]\mathsf{coev}} & [X/Y]^3 & \xrightarrow{[X/Y]^2 \varepsilon_f} & [X/Y]^2 \\ & \searrow^{=} & \downarrow {\scriptstyle \mathsf{TrFm}_f [X/Y]} & & \downarrow {\scriptstyle \mathsf{TrFm}_f} \\ & & [X/Y] & \xrightarrow[\varepsilon_f]{} & 1_{\mathcal{C}_Y} \end{array}$$

For the other case, that of μ_1 one might define dually

$$\mu_1 := 1_{\mathcal{C}_Y} \xrightarrow{\mathsf{coev}} [X/Y]^2 \xrightarrow{\varepsilon_f [X/Y]} [X/Y]$$

This will also work, but the argument proceeds slightly differently. We do however have the dual diagram

$$\begin{array}{ccccc} [X/Y] & \xrightarrow{\mathsf{coev}[X/Y]} & [X/Y]^3 & \xrightarrow{\varepsilon_f [X/Y]^2} & [X/Y]^2 \\ & \searrow^{=} & \downarrow {\scriptstyle [X/Y]\mathsf{TrFm}_f} & & \downarrow {\scriptstyle \mathsf{TrFm}_f} \\ & & [X/Y] & \xrightarrow[\varepsilon_f]{} & 1_{\mathcal{C}_Y} \end{array}$$

from which we see that the composite $\varepsilon_f \circ (f_! \nu_f f^*) \circ (\mu_1 f_! f^*)$ is homotopic to ε_f.

Lemma 4.12. *Consider an adjunction*

$$\mathcal{C} \underset{\mathcal{G}}{\overset{\mathcal{F}}{\rightleftarrows}} \mathcal{D} \quad (\mathcal{F} \dashv \mathcal{G})$$

and let η, ε denote a compatible unit and counit for it. Then there is an equivalence

$$\mathsf{Hom}_{\mathsf{Fun}(\mathcal{C},\mathcal{D})}(\mathcal{F}, \mathcal{F}) \xrightarrow{\mathcal{G}^*} \mathsf{Hom}_{\mathsf{Fun}(\mathcal{D},\mathcal{D})}(\mathcal{F}\mathcal{G}, \mathcal{F}\mathcal{G}) \xrightarrow{\varepsilon_*} \mathsf{Hom}_{\mathsf{Fun}(\mathcal{D},\mathcal{D})}(\mathcal{F}\mathcal{G}, 1_{\mathcal{D}})$$

[9]In fact, μ is homotopic to both μ_1 and μ_2.

[10]This is encapsulated by the fact that the adjunction $f_! \dashv f^*$ induces an adjunction on passing to homotopy categories.

Proof. (of the lemma) This is an instance of the induced adjunction between the precomposition functors. More precisely, for an infinity category $\mathcal{E}$, we have an adjunction

$$\mathcal{E}^{\mathcal{C}} \underset{\mathcal{F}^*}{\overset{\mathcal{G}^*}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathcal{E}^{\mathcal{D}}$$

whose unit and counit can be described in terms of η and ε as $\eta^* \colon 1_{\mathcal{E}^{\mathcal{C}}} \to (\mathcal{G}\mathcal{F})^* = \mathcal{F}^*\mathcal{G}^*$ and $\varepsilon^* \colon \mathcal{G}^*\mathcal{F}^* = (\mathcal{F}\mathcal{G})^* \to 1_{\mathcal{E}^{\mathcal{D}}}$.

Instancing this at $\mathcal{E} := \mathcal{D}$, we have an adjunction equivalence functorial in X, Y

$$\mathsf{Hom}_{\mathsf{Fun}(\mathcal{C},\mathcal{D})}(X, \mathcal{F}^*Y) \xrightarrow{\mathcal{G}^*} \mathsf{Hom}_{\mathsf{Fun}(\mathcal{D},\mathcal{D})}(\mathcal{G}^*X, \mathcal{G}^*\mathcal{F}^*Y) \xrightarrow{(\varepsilon^*)_*} \mathsf{Hom}_{\mathsf{Fun}(\mathcal{D},\mathcal{D})}(\mathcal{G}^*X, Y)$$

which is such that at $X := \mathcal{F}, Y := 1_{\mathcal{D}}$, we have precisely the desired equivalence. □

Instancing the lemma with $\mathcal{F} := f_!, \mathcal{G} := f^*, \eta := \eta_f, \varepsilon := \varepsilon_f$ we see that we have an equivalence

$$\mathsf{Hom}_{\mathsf{Fun}(\mathcal{C}_X,\mathcal{C}_Y)}(f_!, f_!) \xrightarrow{(f^*)^*} \mathsf{Hom}_{\mathsf{Fun}(\mathcal{C}_Y,\mathcal{C}_Y)}(f_!f^*, f_!f^*) \xrightarrow{(\varepsilon_f)_*} \mathsf{Hom}_{\mathsf{Fun}(\mathcal{C}_Y,\mathcal{C}_Y)}(f_!f^*, 1_{\mathcal{C}_Y})$$

Under which $(f_!\nu_f) \circ (\mu_1 f_!)$ and the identity correspond to $\varepsilon_f \circ (f_!\nu_f f^*) \circ (\mu_1 f_! f^*)$ and ε_f respectively. As in the other case, passing to homotopy categories establishes that $(f_!\nu_f) \circ (\mu_1 f_!)$ is homotopic to the identity, completing the proof. □

Corollary 4.13. *For an $(n-1)$-semiadditive infinity category $\mathcal{C}$ with n-finite colimits, $\mathcal{C}$ is n-semiadditive if and only if for every n-finite space X, the Trace Form TrFm_X exhibits $[X/*]$ as self dual in $\mathsf{Fun}(\mathcal{C},\mathcal{C})$.*

Definition 4.14. For an n-finite space X, let Tr_X be the morphism of $\mathcal{S}_n^{n-1}$ given by the span

$$X \times X \xleftarrow{\delta_X} X \xrightarrow{p} *$$

4.3. **Statement of the criterion.** We can now state precisely the actual criterion we will devote most of this section to proving.

Proposition 4.15. *(cf.* [Har20, Propostion 3.17]*) Let $\mathcal{C}$ be an infinity category satisfying the standing hypothesis (Hypothesis 4.3). Then, $\mathcal{C}$ is n-semiadditive if and only if for every n-finite space X, the transformation*

$$[X] \circ [X] \simeq [X \times X] \xrightarrow{[\mathsf{Tr}_X]} 1_{\mathcal{C}}$$

exhibits $[X]$ as self dual in $\mathsf{Fun}(\mathcal{C},\mathcal{C})$.

The connection between this criterion and an $\mathcal{S}_n^n$-action is due to the fact that such an action lets us explicitly write out a coevaluation compatible with the transformation just described.

Lemma 4.16. *Let X be an n-finite space, then the trace map $X \times X \xrightarrow{\mathsf{Tr}_X} *$ and its dual $* \xrightarrow{\widehat{\mathsf{Tr}_X}} X \times X$ in $\mathcal{S}_n^n$ are an evaluation-coevaluation pair exhibiting X as self dual.*

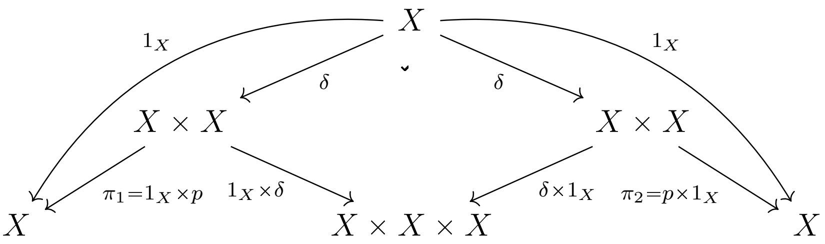
Proof. What this means is to check that the pairs of composable maps in $\mathcal{S}_n^n$

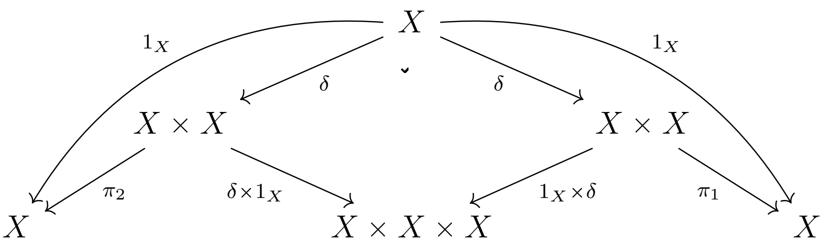
$$X \xrightarrow{1_X \times \widehat{Tr_X}} X \times X \times X \xrightarrow{\mathsf{Tr}_X \times 1_X} X$$

$$X \xrightarrow{\widehat{Tr_X} \times 1_X} X \times X \times X \xrightarrow{1_X \times \mathsf{Tr}_X} X$$

have 1_X as a composition.

We have the diagrams of (n-finite) spaces (where δ is the diagonal of X and p is the map to the point)





which witness precisely compositions as such. ☐

Corollary 4.17. *Consider an infinity category $\mathcal{C}$ with n-finite colimits, and an action of $\mathcal{S}_n^n$ such that the action functor preserves n-finite colimits variable-wise. $\mathcal{C}$ is necessarily n-semiadditive.*

Proof. As in the discussion at the start of this section, we proceed by induction on n, the base case $n = -2$ being tautological. As the restriction of the action to $\mathcal{S}_{n-1}^{n-1}$ preserves $(n-1)$-finite colimits variable-wise, we may inductively assume that $\mathcal{C}$ is $(n-1)$-semiadditive. In particular, it satisfies the standing hypothesis.

Thus it suffices to show that the trace form

$$[X] \circ [X] \simeq [X \times X] \xrightarrow{[\mathsf{Tr}_X]} 1_{\mathcal{C}}$$

exhibits $[X]$ as self dual, that is, it is an evaluation map. But in $\mathcal{S}_n^n$, $X \times X \xrightarrow{\mathsf{Tr}_X} *$ and its dual span $* \xrightarrow{\widehat{\mathsf{Tr}_X}} X \times X$ are an evaluation-coevaluation pair. This remains true on passing to their actions, proving self-duality as required. ☐

Example 4.18. For $-2 \leqslant m \leqslant n$, the infinity category $\mathcal{S}_n^m$ is m-semiadditive. This is an immediate consequence of the above corollary, as we have seen that $\mathcal{S}_n^m$ has m-finite colimits (n-finite, even) and that the monoidal product on $\mathcal{S}_n^m$ restricts to an action of $\mathcal{S}_m^m$ on $\mathcal{S}_n^m$ that preserves m-finite colimits variable-wise.

The strategy to prove Proposition 4.15 is to identify the described transformation with the TrFm_X of Hopkins and Lurie. The proof will ultimately involve finding equivalent expressions of the various factors, and will be done in several steps.

Proposition 4.19. *Consider an infinity category $\mathcal{C}$ satisfying the standing hypothesis (Hypothesis 4.3)and an n-finite space X. Denote by $X \xrightarrow{p} *$ the map to the terminal object (in $\mathcal{S}_n$), then the induced*

$$p_!p^* \simeq [X] \xrightarrow{[p]} [*] \simeq 1_{\mathcal{C}}$$

is a counit of the adjunction $p_! \dashv p^*$.

Proof. To see that $[p]$ is a counit to the (a-priori known to exist) adjunction $p_! \dashv p^*$, it suffices to show that its adjoint $\widetilde{[p]} \colon p^* \to p^*$ is invertible. This is computed by applying the functor p^* and precomposing by its unit.

The adjunction $p_! \dashv p^*$ in question is an adjunction $\mathsf{colim}_X \dashv (-)_X$, and the unit is the $(-) \to (\mathsf{colim}_X -)_X$, given by the colimit cocone maps. We have already calculated this to be the transformation $[i_x]_{x \in X}$ whose component at an $x \in X$ is the action of the canonical colimit cocone map i_x of Construction 2.12.

Thus, the adjoint $\widetilde{[p]}$ is such that its evaluation at any $x \in X$ is given by the composition $[p] \circ [i_x] \simeq [p \circ i_x] \simeq 1_{1_\mathcal{C}}$ (as i_x is a section of p). In particular, it is object-wise an equivalence in $\mathsf{Fun}(\mathcal{C}, \mathcal{C}^X) \cong \mathsf{Fun}(X, \mathsf{Fun}(\mathcal{C}, \mathcal{C}))$, and hence an equivalence as desired. □

Proposition 4.20. *Consider an infinity category* $\mathcal{C}$ *satisfying the standing hypothesis (Hypothesis 4.3), and an* $(n-1)$*-finite space* X*. Denoting by* $p \colon X \to *$ *the map to the point, the induced*

$$1_\mathcal{C} \simeq [*] \xrightarrow{[\widehat{p}]} [X] \simeq p_! p^*$$

is a unit for an adjunction $p^* \dashv p_!$ *, that is, exhibits* X *as* $\mathcal{C}$*-ambidextrous.*

Proof. As $\mathcal{C}$ is $(n-1)$-semiadditive, the inductive construction of [HopLur13] constructs a ν_X^k that is a counit of an adjunction $p^* \dashv p_!$. Let μ_X^k be a unit compatible with it, we will show that $\mu_X^k \simeq [\widehat{p}]$.

Proposition 4.11 (that is, [HopLur13, Proposition 5.1.8]), which characterizes ambidexterity in terms of the trace form asserts that

$$\mathsf{TrFm}_X \colon \; [X] \circ [X] \simeq p_! p^* p_! p^* \xrightarrow{p_! \nu_X^k p^*} p_! p^* \xrightarrow{\varepsilon_X} 1_\mathcal{C}$$

exhibits $[X]$ as self dual in $\mathsf{Fun}(\mathcal{C}, \mathcal{C})$. Here, ε_X is a counit to the adjunction $p_! \dashv p^*$. We know from Proposition 4.19 that $[p]$ is such a counit, so we may as well set $\varepsilon_X = [X]$.

To identify the maps $[\widehat{p}]$ and μ_X^k in the monoidal category $\mathsf{Fun}(\mathcal{C}, \mathcal{C})$, it suffices to identify their duals. The dual of $\widehat{p}$ is just p, so the dual of $[\widehat{p}]$ is $[p]$ as taking the action is compatible with duals. The dual of μ_X^k is computed as

$$[X] \simeq [X] \circ 1_\mathcal{C} \xrightarrow{[X]\mu_X^k} [X] \circ [X] \xrightarrow{\mathsf{TrFm}_X} 1_\mathcal{C}$$

which expands as

$$p_! p^* \xrightarrow{p_! p^* \mu_X^k} p_! p^* p_! p^* \xrightarrow{p_! \nu_X^k p^*} p_! p^* \xrightarrow{[p]} 1_\mathcal{C}$$

The first two maps in the sequence are just $p_!$ applied to the two composable maps in one of the triangle identities for $p^* \dashv p_!$. Thus, $[p]$ is a composition of the entire sequence, that is, $[p]$ is a dual of μ_X^k as well. As μ_X^k and $[\widehat{p}]$, have the same duals, they are equivalent, and consequently $[\widehat{p}]$ is a unit for an adjunction as desired. □

So far we have been concerned with the action of objects, that is the endomorphisms $p_! p^*$ of $\mathcal{C}$ induced by a $p \colon X \to *$. We now extend this to a description of $f_! f^*$ for more general $f \colon X \to Y$.

We expect this to be a functor $\mathcal{C}^Y \to \mathcal{C}^Y$. One might expect this to derive from an action on $\mathcal{C}^Y$. This can be made precise using the straightening equivalence.

Definition 4.21. Recall that the straightening equivalence (cf. [Cis19, Corollary 6.5.9])

$$\mathsf{St}_X \colon \mathcal{S}_{/X} \xrightarrow{\sim} \mathsf{Fun}(X, \mathcal{S})$$

restricts to a

$$\mathsf{St}_X \colon {\mathcal{S}_n}_{/X} \to \mathsf{Fun}(X, \mathcal{S}_n)$$

for any n-finite space X[11] (by a Serre long exact sequence argument).

Further, we denote the composite induced by $\mathcal{S}_n \hookrightarrow \mathcal{S}_n^{n-1}$ as

$$\overline{\mathsf{St}}_X \colon {}^{\mathcal{S}_n}\!/_X \to \mathsf{Fun}(X, \mathcal{S}_n) \to \mathsf{Fun}(X, \mathcal{S}_n^{n-1}) = \left(\mathcal{S}_n^{n-1}\right)^X$$

(or simply $\bar{\mathsf{St}}$ when X is clear from context)

Notation 4.22. For a map $f \colon X \to Y$ in $\mathcal{S}_n$, we denote by $\mathsf{St}_f \colon Y \to \mathcal{S}_n$ its straightening, and by $\overline{\mathsf{St}}_f \colon Y \to \mathcal{S}_n^{n-1}$ the further composite with the inclusion into $\mathcal{S}_n^{n-1}$, which computes the image of f under $\bar{\mathsf{St}}$.

Construction 4.23. For an infinity category $\mathcal{C}$ satisfying the standing hypothesis (Hypothesis 4.3), the action of $\mathcal{S}_n^{n-1}$ on $\mathcal{C}$ induces for every n-finite space Y an action of $\left(\mathcal{S}_n^{n-1}\right)^Y$ on $\mathcal{C}^Y$ point-wise. As colimits in functor categories are computed object-wise, this action also preserves n-finite colimits in each variable.

For an $f \colon X \to Y$, the action of $\overline{\mathsf{St}}_f$ on an $Y \xrightarrow{\mathcal{F}} \mathcal{C}$ can be explicitly computed as

$$\left(\left[\overline{\mathsf{St}}_f\right]\mathcal{F}\right)_y = \left[\overline{\mathsf{St}}_f(y)\right](\mathcal{F}_y) \simeq [X_y](\mathcal{F}_y)$$

Proposition 4.24. *For an infinity category $\mathcal{C}$ satisfying the standing hypothesis (Hypothesis 4.3), and a $f \colon X \to Y$ of n-finite spaces, we have an identification*

$$\left[\overline{\mathsf{St}}_f\right] \simeq f_! f^* \colon \mathcal{C}^Y \to \mathcal{C}^Y$$

Proof. We describe a map $f_! f^* \to \left[\overline{\mathsf{St}}_f\right]$ that will be shown to be an equivalence. Form the base change

$$\begin{array}{ccc} X \times_Y X & \xrightarrow{g'} & X \\ {\scriptstyle g}\downarrow & \lrcorner & \downarrow{\scriptstyle f} \\ X & \xrightarrow[f]{} & Y \end{array}$$

and let $\delta \colon X \to X \times_Y X$ be the diagonal of f. This is a morphism in the slice ${}^{\mathcal{S}_n}\!/_X$ (where $X \times_Y X$ is given structure map g), and thus defines a morphism $*_X \simeq \mathsf{St}_{1_X} \xrightarrow{\delta} \mathsf{St}_g$. As straightening is compatible with base change, we can identify $\mathsf{St}_g \simeq f^* \mathsf{St}_f$. More explicitly, we have a(n essentially) commutative diagram

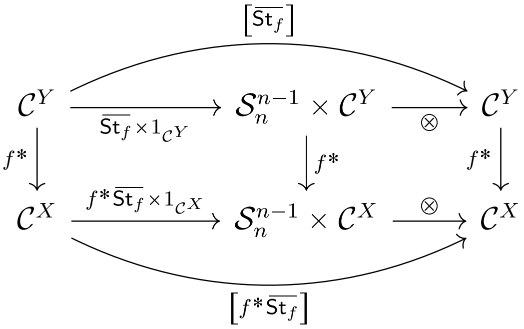
$$\begin{array}{ccc} {}^{\mathcal{S}_n}\!/_Y & \xrightarrow{\sim} & \mathsf{Fun}(Y, \mathcal{S}_n) \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle f^*} \\ {}^{\mathcal{S}_n}\!/_X & \xrightarrow[\sim]{} & \mathsf{Fun}(X, \mathcal{S}_n) \end{array}$$

so that the two images of f are identified.

Consequently, one gets a map $1_{\mathcal{C}^X} \xrightarrow{[\delta]} \left[\overline{\mathsf{St}}_g\right] \simeq \left[f^* \overline{\mathsf{St}}_f\right]$ in $\mathsf{Fun}(\mathcal{C}^X, \mathcal{C}^X)$. Precomposing with f^* provides a $f^* \xrightarrow{[\delta] f^*} \left[f^* \overline{\mathsf{St}}_f\right] f^*$ in $\mathsf{Fun}(\mathcal{C}^Y, \mathcal{C}^X)$. This last term is itself just

[11] And further in fact, to a map on the m-truncated maps ${}^{\mathcal{S}_{n,m}}\!/_X \to \mathsf{Fun}(X, \mathcal{S}_m)$

$f^*\left[\overline{\mathsf{St}_f}\right]$, in light of the diagram



The map $f^* \xrightarrow{[\delta] f^*} f^*\left[\overline{\mathsf{St}_f}\right]$ is adjoint to a $f_! f^* \to \left[\overline{\mathsf{St}_f}\right]$, which we will show is an equivalence.

This can be done point-wise in $\mathcal{F} \in \mathcal{C}^Y$. We must show that $f^*\mathcal{F} \xrightarrow{[\delta] f^*\mathcal{F}} f^*\left[\overline{\mathsf{St}_f}\right]\mathcal{F}$ exhibits $\left[\overline{\mathsf{St}_f}\right] : Y \to \mathcal{C}$ as the left Kan-extension of $f^*\mathcal{F} = \mathcal{F} \circ f \colon X \to \mathcal{C}$ along f. By the formula for constructing left Kan-extensions along maps between spaces of Lemma 2.17, it suffices to check that for every $y \in Y$, the map exhibits

$$\left(\left[\overline{\mathsf{St}_f}\right]\mathcal{F}\right)_y \simeq [X_y](\mathcal{F}_y) \simeq \operatorname{colim}(f^*\mathcal{F})|_{X_y}$$

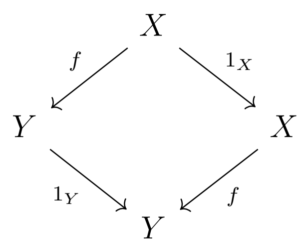
in $\mathcal{C}$, or in other words, that

$$(\mathcal{F}_y)_{X_y} = (f^*\mathcal{F})|_{X_y} \xrightarrow{[\delta]} ([X_y]\,\mathcal{F}_y)_{X_y}$$

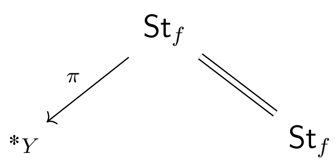
is a colimit cocone (in $\mathsf{Fun}(X_y, \mathcal{C})$).

To see this, note that the restriction of $[\delta]$ to X_y is given by the map induced by $[-]$ on the restriction of δ to X_y. But this was defined by applying the straightening construction to $X \to X \times_Y X$ over X. Its restriction to X_y is thus given by straightening the map given by pulling $X \to X \times_Y X$ to a map over X_y. This map is simply the diagonal $X_y \to X_y \times X_y$ of X_y. We have already seen that the straightening of this map is just the canonical cocone $(i_x)_{x \in X_y}$ of Construction 2.12. Thus, upon taking the action on $\mathcal{F}_y \in \mathcal{C}$, we get precisely the colimit cone as desired, proving the claim. □

Construction 4.25. Consider a morphism $f\colon X \to Y$ in $\mathcal{S}_{n,(n-1)}$. We have the span in $\mathcal{S}_n \diagup_Y$.



Straightening converts this into a span in $\mathsf{Fun}(Y, \mathcal{S}_n)$



The wrong way map is $(n-1)$-truncated. Recall from Proposition 1.26 that forming spans is compatible with exponentials. Thus this is equivalently a morphism in $(\mathcal{S}_n^{n-1})^Y \simeq \mathsf{Span}\left(\mathcal{S}_n^Y, \mathcal{S}_{n,(n-1)}^Y\right)$ which we denote as $\left(\widehat{f}\right)_Y : *_Y \to \overline{\mathsf{St}}_f$.

Definition 4.26. (cf. [Har20, discussion before Lemma 3.11]) For an infinity category $\mathcal{C}$ satisfying the standing hypothesis (Hypothesis 4.3), and a map $f\colon X \to Y$ in $\mathcal{S}_{n,(n-1)}$, denote the transformation the action of $\left(\widehat{f}\right)_Y$ induces by

$$\left[\widehat{f}\right]_Y : 1_{\mathcal{C}^Y} \to \left[\overline{\mathsf{St}}_f\right] \simeq f_! f^*$$

Proposition 4.27. *(cf. [Har20, Lemma 3.12]) For an infinity category $\mathcal{C}$ satisfying the standing hypothesis (Hypothesis 4.3), and a map $f\colon X \to Y$ in $\mathcal{S}_{n,(n-1)}$, the so defined transformation $\left[\widehat{f}\right]_Y : 1_{\mathcal{C}^Y} \to f_! f^*$ is a unit for an adjunction $f^* \dashv f_!$, that is, exhibits f as $\mathcal{C}$-ambidextrous.*

Proof. We must show that for $X \xrightarrow{\mathcal{F}} \mathcal{C}, Y \xrightarrow{\mathcal{G}} \mathcal{C}$ the composite

$$\mathsf{Hom}_{\mathcal{C}^X}\left(f^*\mathcal{G}, \mathcal{F}\right) \xrightarrow{(f_!)_{f^*\mathcal{G},\mathcal{F}}} \mathsf{Hom}_{\mathcal{C}^Y}(f_! f^*\mathcal{G}, f_!\mathcal{F}) \xrightarrow{\left([\hat{f}]_Y \mathcal{G}\right)^*} \mathsf{Hom}_{\mathcal{C}^Y}(\mathcal{G}, f_!\mathcal{F})$$

is an equivalence. This is a family of maps functorial in $\mathcal{F}, \mathcal{G}$ and compatible with colimits in the $\mathcal{G}$-variable. We have seen in Lemma 3.17 that $\mathsf{Fun}(Y, \mathcal{C})$ is generated under Y-indexed colimits, so it suffices to consider the case where $\mathcal{G}$ is $y_! C$, for an object C of $\mathcal{C}$ and an object $* = \{y\} \xrightarrow{y} Y$ of Y.

Remark 4.28. Assuming for the moment that $\left[\widehat{f}\right]_Y$ is indeed a unit for $f^* \dashv f_!$, then for any $y \in Y$ the composite

$$1_{\mathcal{C}} \xrightarrow{\eta^y} y^* y_! \xrightarrow{y^*[\hat{f}]_Y y_!} y^* f_! f^* y_!$$

is a unit for the composite adjunction $f^* y_! \dashv y^* f_!$, where η^y is a unit for $y_! \dashv y^*$.

Lemma 4.29. *The converse is true. That is, if for every object $y \in Y$, the composite*

$$\lambda\colon 1_{\mathcal{C}} \xrightarrow{\eta^y} y^* y_! \xrightarrow{y^*[\hat{f}]_Y y_!} y^* f_! f^* y_!$$

is a unit for an adjunction $f^* y_! \dashv y^* f_!$, then $\left[\widehat{f}\right]_Y$ is a unit for an adjunction $f^* \dashv f_!$.

Proof. (of the lemma) By the colimit generation property, as remarked above it suffices to show that for every $y \in Y$ and $C \in \mathcal{C}$, the composite

$$\mathsf{Hom}_{\mathcal{C}^X}\left(f^* y_! C, \mathcal{F}\right) \xrightarrow{(f_!)_{f^* y_! C,\mathcal{F}}} \mathsf{Hom}_{\mathcal{C}^Y}(f_! f^* y_! C, f_!\mathcal{F}) \xrightarrow{\left([\hat{f}]_Y y_! C\right)^*} \mathsf{Hom}_{\mathcal{C}^Y}(y_! C, f_!\mathcal{F})$$

is an equivalence. For this, note that there is a commutative square

$$\begin{array}{ccc} \mathsf{Hom}_{\mathcal{C}^Y}\left(f_! f^* y_! C, f_!\mathcal{F}\right) & \xrightarrow{(y^*)_{f_! f^* y_! C, f_!\mathcal{F}}} & \mathsf{Hom}_{\mathcal{C}}\left(y^* f_! f^* y_! C, f_! y^*\mathcal{F}\right) \\ \left([\hat{f}]_Y y_! C\right)^* \big\downarrow & & \big\downarrow \left(y^*[\hat{f}]_Y^* y_! C\right)^* \\ \mathsf{Hom}_{\mathcal{C}^Y}\left(y_! C, f_!\mathcal{F}\right) & \xrightarrow{(y^*)_{y_! C, f_!\mathcal{F}}} & \mathsf{Hom}_{\mathcal{C}}\left(y^* y_! C, f_! y^*\mathcal{F}\right) \end{array}$$

fitting into a diagram

$$\begin{array}{ccccc}
\mathsf{Hom}_{\mathcal{C}^X}\left(f^* y_! C, \mathcal{F}\right) & \xrightarrow{(f_!)_{f^* y_! C, \mathcal{F}}} & \mathsf{Hom}_{\mathcal{C}^Y}\left(f_! f^* y_! C, f_! \mathcal{F}\right) & \xrightarrow{(y^*)_{f_! f^* y_! C, f_! \mathcal{F}}} & \mathsf{Hom}_{\mathcal{C}}\left(y^* f_! f^* y_! C, f_! y^* \mathcal{F}\right) \\
& \searrow^{\lambda} & \downarrow{\scriptstyle \left(\left[\widehat{f}\right]_Y y_! C\right)^*} & & \downarrow{\scriptstyle \left(y^* \left[\widehat{f}\right]^*_Y y_! C\right)^*} \\
& & \mathsf{Hom}_{\mathcal{C}^Y}\left(y_! C, f_! \mathcal{F}\right) & \xrightarrow{(y^*)_{y_! C, f_! \mathcal{F}}} & \mathsf{Hom}_{\mathcal{C}}\left(y^* y_! C, f_! y^* \mathcal{F}\right) \\
& & & \searrow^{\sim} & \downarrow{\scriptstyle \eta^y} \\
& & & & \mathsf{Hom}_{\mathcal{C}}(C, f_! y^* \mathcal{F})
\end{array}$$

The total composite $\mathsf{Hom}_{\mathcal{C}^X}\left(f^* y_! C, \mathcal{F}\right) \to \mathsf{Hom}_{\mathcal{C}}(C, f_! y^* \mathcal{F})$ is just the adjunction map for $f^* y_! \dashv f_! y^*$ and hence invertible. The map $\mathsf{Hom}_{\mathcal{C}^Y}\left(y_! C, f_! \mathcal{F}\right) \to \mathsf{Hom}_{\mathcal{C}}(C, f_! y^* \mathcal{F})$ is the adjunction map for $y_! \dashv y^*$ and is thus invertible as well. Consequently, the map λ is invertible. The invertiblity of λ for arbitrary y and C is precisely what we need to conclude the lemma. □

To see that the composite λ in the hypothesis of the lemma is indeed a unit, consider for arbitrary $y \in Y$ and $C \in \mathcal{C}$ the pullback square

$$\begin{array}{ccc}
X_y & \xrightarrow{\iota_y} & X \\
\downarrow{\scriptstyle f_y} & \lrcorner \;\; \sigma & \downarrow{\scriptstyle f} \\
\{y\} & \xrightarrow[y]{} & Y
\end{array}$$

The identification $y^* \left[\overline{\mathsf{St}}_f\right] \simeq \left[y^* \overline{\mathsf{St}}_f\right] y^* \simeq \left[X_y\right] y^*$ can be written in terms of the Beck-Chevalley transformation $\mathsf{BC}\left[\sigma\right] : \left(f_y\right)_! \iota_y^* \to y^* f_!$. Explicitly, the inverse of $\mathsf{BC}\left[\sigma\right]$ defines the equivalence as

$$y^* \left[\overline{\mathsf{St}}_f\right] \simeq y^* f_! f^* \xrightarrow[\sim]{\mathsf{BC}[\sigma]^{-1} f^*} \left(f_y\right)_! \iota_y^* f^* \simeq \left(f_y\right)_! f_y^* y^* \simeq \left[X_y\right] y^*$$

and in particular it is such that

$$\begin{array}{ccc}
& y^* & \\
{\scriptstyle y^* \left[\widehat{f}\right]_Y} \swarrow & & \searrow {\scriptstyle \left[\widehat{f_y}\right] y^*} \\
y^* \left[\overline{\mathsf{St}}_f\right] & \xrightarrow{\sim} & \left[X_y\right] y^*
\end{array}$$

Thus the second map in the composition defining λ, $y^* \left[\widehat{f}\right]_Y y_!$ can be identified with the composite

$$y^* y_! \xrightarrow{\left[\widehat{f_y}\right] y^* y_!} \left(f_y\right)_! \left(f_y\right)^* y^* y_! \simeq \left(f_y\right)_! \left(\iota_y\right)^* f^* y_! \xrightarrow{\mathsf{BC}[\sigma] f^* y^*} y^* f_! f^* y_!$$

In particular, this fits into a diagram

$$\begin{array}{ccccc}
1_{\mathcal{C}} & \xrightarrow{\eta^y} & y^* y_! & \xrightarrow{y^* \left[\widehat{f}\right]_Y} & y^* f_! f^* y_! \\
\downarrow{\scriptstyle \left[\widehat{f_y}\right]} & & \downarrow{\scriptstyle \left[\widehat{f_y}\right] y^* y_!} & & \uparrow{\scriptstyle \mathsf{BC}[\sigma] f^* y^*} \\
\left(f_y\right)_! \left(f_y\right)^* & \xrightarrow[\left(f_y\right)_! f_y^* \eta^y]{} & \left(f_y\right)_! \left(f_y\right)^* y^* y_! & \xrightarrow[\sim]{} & \left(f_y\right)_! \left(\iota_y\right)^* f^* y_!
\end{array}$$

and so equivalently we must show that the composite

$$\varphi \colon 1_{\mathcal{C}} \xrightarrow{\left[\widehat{f_y}\right]} \left(f_y\right)_! \left(f_y\right)^* \xrightarrow{\left(f_y\right)_! \left(f_y\right)^* \eta^y} \left(f_y\right)_! \left(f_y\right)^* y^* y_! \simeq \left(f_y\right)_! \left(\iota_y\right)^* f^* y_! \xrightarrow{\mathsf{BC}[\sigma] f^* y_!} y^* f_! f^* y_!$$

is a unit for $f^* y_! \dashv y^* f_!$.

To do so, recall that $\mathsf{BC}[\sigma]$ is defined as the adjoint of $y_!\,(f_y)_!\,\iota_y^* \simeq f_!\,(\iota_y)_!\,\iota_y^* \xrightarrow{f_!\varepsilon^{\iota_y}} f_!$, where $\eta^{\iota_y}, \varepsilon^{\iota_y}$ are unit and counit for $(\iota_y)_! \dashv \iota_y^*$. Explicitly, this is[12]

$$(f_y)_!\,\iota_y^* \xrightarrow{\eta^y (f_y)_! \iota_y^*} y^* y_!\,(f_y)_!\,\iota_y^* \simeq y^* f_!\,(\iota_y)_!\,\iota_y^* \xrightarrow{y^* f_! \varepsilon^{\iota_y}} y^* f_!$$

Similarly, the Beck-Chevalley map of the transpose σ^t of the pullback square σ is given by

$$(\iota_y)_!\,f_y^* \xrightarrow{(\iota_y)_! f_y^* \eta^y} (\iota_y)_!\,f_y^* y^* y_! \simeq (\iota_y)_!\,\iota_y^* f^* y_! \xrightarrow{\varepsilon^{\iota_y} f^* y_!} f^* y_!$$

There is a commutative diagram

$$\begin{array}{ccccccc}
f_y^* & \xrightarrow{f_y^* \eta^y} & f_y^* y^* y_! & \xrightarrow{\sim} & \iota_y^* f^* y_! & = & \iota_y^* f^* y_! \\
\downarrow{\scriptstyle \eta^{\iota_y} f_y^*} & & \downarrow{\scriptstyle \eta^{\iota_y} f_y^* y^* y_!} & & \downarrow{\scriptstyle \eta^{\iota_y} \iota_y^* f^* y_!} & & \Vert \\
\iota_y^*\,(\iota_y)_!\,f_y^* & \xrightarrow{\iota_y^* (\iota_y)_! f_y^* \eta^y} & \iota_y^*\,(\iota_y)_!\,f_y^* y^* y_! & \xrightarrow{\sim} & \iota_y^*\,(\iota_y)_!\,\iota_y^* f^* y_! & \xrightarrow{\iota_y^* \varepsilon^{\iota_y} f^* y_!} & \iota_y^* f^* y_!
\end{array}$$

whose bottom row is precisely $\iota_y^* \mathsf{BC}[\sigma^t]$. Consequently, upon applying $(f_y)_!$, this can be seen to fit into

$$\begin{array}{ccccccc}
1_{\mathcal{C}} & \xrightarrow{[\widehat{f_y}]} & (f_y)_!\,f_y^* & \xrightarrow{(f_y)_! \eta^{\iota_y} f_y^*} & (f_y)_!\,\iota_y^*\,(\iota_y)_!\,f_y^* & & \\
 & & \downarrow{\scriptstyle (f_y)_! f_y^* \eta^y} & & \downarrow{\scriptstyle (f_y)_! \iota_y^* \mathsf{BC}[\sigma^t]} & \searrow{\scriptstyle \mathsf{BC}[\sigma]\mathsf{BC}[\sigma^t]} & y^* f_! f^* y_! \\
 & & (f_y)_!\,f_y^* y^* y_! & \xrightarrow{\sim} & (f_y)_!\,\iota_y^* f^* y_! & \nearrow{\scriptstyle \mathsf{BC}[\sigma] f^* y_!} &
\end{array}$$

and thus the composite λ (equivalently, φ) can be further identified with

$$\phi\colon 1_{\mathcal{C}} \xrightarrow{[\widehat{f_y}]} (f_y)_!\,f_y^* \xrightarrow{(f_y)_! \eta^{\iota_y} f_y^*} (f_y)_!\,\iota_y^*\,(\iota_y)_!\,f_y^* \xrightarrow{\mathsf{BC}[\sigma]\mathsf{BC}[\sigma^t]} y^* f_! f^* y_!$$

Lemma 4.30. *Consider an adjunction*

$$\mathcal{C} \underset{\mathcal{G}}{\overset{\mathcal{F}}{\underset{\perp}{\rightleftarrows}}} \mathcal{D}$$

and equivalences $\mathcal{F} \xrightarrow{\alpha} \mathcal{F}', \mathcal{G} \xrightarrow{\beta} \mathcal{G}'$. *Then, if* η, ε *are unit and counit for* $\mathcal{F} \dashv \mathcal{G}$, *the composites*

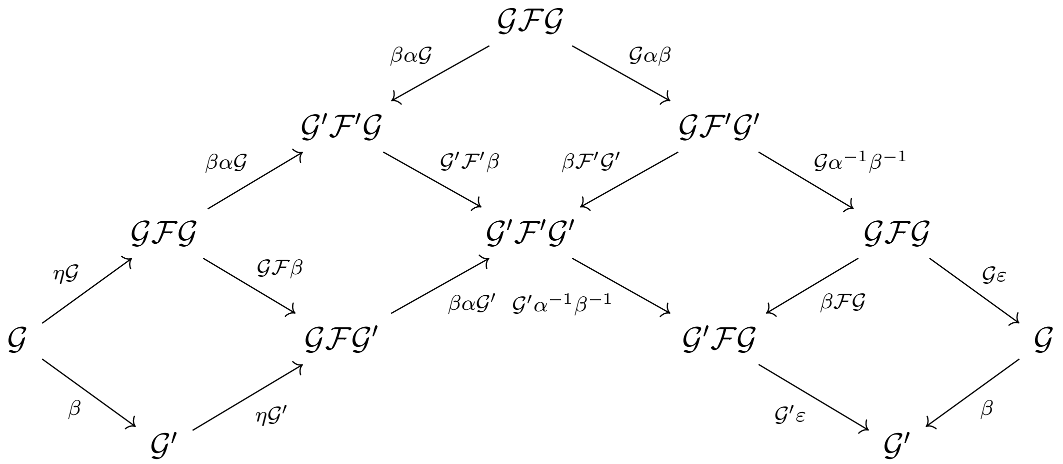
$$1_{\mathcal{C}} \xrightarrow{\eta} \mathcal{G}\mathcal{F} \underset{\sim}{\xrightarrow{\beta\alpha}} \mathcal{G}'\mathcal{F}'$$

$$\mathcal{F}'\mathcal{G}' \underset{\sim}{\xrightarrow{\alpha^{-1}\beta^{-1}}} \mathcal{F}\mathcal{G} \xrightarrow{\varepsilon} 1_{\mathcal{D}}$$

are unit and counit for a $\mathcal{F}' \dashv \mathcal{G}'$.

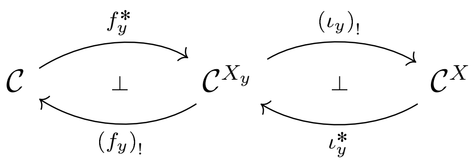
[12]Or equivalently, $(f_y)_!\,\iota_y^* \xrightarrow{(f_y)_! \iota_y^* \eta^f} (f_y)_!\,\iota_y^* f^* f_! \simeq (f_y)_!\,f_y^* y^* f_! \xrightarrow{\varepsilon^{f_y} y^* f_!} y^* f_!$

Proof. (of the lemma) One checks the triangle identities. We have the diagram



from which one sees that the precomposition of the triangle map for $\mathcal{G}'$ with β is the postcomposition of β with the triangle map of $\mathcal{G}$, that is, β itself. As β is invertible, one concludes that $1_{\mathcal{G}'}$ is a composition of the triangle map as required. A similar argument demonstrates the triangle identity for $\mathcal{F}'$. $\square$

Instancing the lemma with $\alpha = \mathsf{BC}[\sigma^t], \beta = \mathsf{BC}[\sigma]$ (and the other variables instanced so that this makes sense), it therefore in fact suffices to show that the composite of the first two maps in the composition defining ϕ (call this composition ψ) is a unit for an adjunction $(\iota_y)_! f_y^* \dashv (f_y)_! \iota_y^*$. However, we have seen in Proposition 4.20 that $\left[\widehat{f_y}\right]$ is a unit for $f_y^* \dashv (f_y)_!$. Thus this map ψ is just the composite unit for the adjunctions



and is thus a unit as desired, proving the claim. $\square$

For an infinity category $\mathcal{C}$ satisfying the standing hypothesis (Hypothesis 4.3)and a map $f\colon X \to Y$ between n-finite spaces, we have already seen that f is weakly ambidextrous, in particular there is a ν_f^k produced by the inductive construction of [HopLur13].

This is defined in terms of a μ_δ^k, which is a unit for an adjunction $\delta^* \dashv \delta_!$ (where $\delta\colon X \to X \times_Y X$ is the diagonal of f).

But Proposition 4.27 above establishes that the transformation $\left[\widehat{\delta}\right]_Y$ is also a unit for such an adjunction, and in particular homotopic to μ_δ^k. Our approach will be to use $\left[\widehat{\delta}\right]_Y$ as a replacement for μ_δ^k to define a transformation ν_f in an identical manner, which can be identified with the ν_f^k of [HopLur13].

Definition 4.31. Consider an infinity category $\mathcal{C}$ satisfying the standing hypothesis (Hypothesis 4.3)and a morphism $f\colon X \to Y$ in $\mathcal{S}_n$. Denote by δ_f the diagonal of f. Then we define a transformation ν_f by forming the pullback square σ

$$\begin{array}{ccc} X \times_Y X & \xrightarrow{\pi_2} & X \\ \big\downarrow{\scriptstyle \pi_1} & \lrcorner \quad \sigma & \big\downarrow{\scriptstyle f} \\ X & \xrightarrow[f]{} & Y \end{array}$$

and taking the composite

$$\nu_f\colon f^*f_! \xrightarrow[\sim]{\mathsf{BC}[\sigma]^{-1}} \pi_{1!}\pi_2^* \xrightarrow{\pi_{1!}\left[\widehat{\delta_f}\right]_{X\times_Y X}\pi_2^*} \pi_{1!}\delta_{f!}\delta_f^*\pi_2^* \simeq 1_{\mathcal{C}_X}\circ 1_{\mathcal{C}_X} \simeq 1_{\mathcal{C}_X}$$

Observation 4.32. *Consider an infinity category $\mathcal{C}$ satisfying the standing hypothesis (Hypothesis 4.3). For any morphism f between n-finite spaces, the transformation ν_f is equivalent to the transformation ν_f^k produced by the construction of* [HopLur13].

In particular, a morphism f in $\mathcal{S}_n$ is $\mathcal{C}$-ambidextrous if and only if the transformation $\nu_f\colon f^*f_! \to 1_{\mathcal{C}_X}$ is a counit of an adjunction $f^* \dashv f_!$.

Proposition 4.33. *(cf.* [Har20, Lemma 3.11]*) For an infinity category $\mathcal{C}$ satisfying the standing hypothesis (Hypothesis 4.3), and a map $f\colon X \to Y$ in $\mathcal{S}_{n,(n-1)}$, the transformation $[Y] \xrightarrow{[\hat{f}]} [X]$ induced by the dual span $\hat{f}$ to f in $\mathcal{S}_n^{n-1}$ can be identified with the composite*

$$[Y] \simeq q_!q^* \xrightarrow{q_!\left[\hat{f}\right]_Y q^*} q_!f_!f^*q^* \simeq p_!p^* \simeq [X]$$

where p and q are the structure maps as in the following diagram.

$$\begin{array}{ccccc} X & & \xrightarrow{\quad f \quad} & & Y \\ & {\scriptstyle p}\searrow & & \swarrow{\scriptstyle q} & \\ & & Z & & \end{array}$$

Proof. The first observation is that we can use the fact that the action functor in question, $[-] : \mathcal{S}_n^{n-1} \to \mathsf{Fun}(\mathcal{C},\mathcal{C})$ computes with n-finite colimits. Applying it to $q_! = \mathsf{colim}_Y$ for instance, we rewrite $q_!\left[\hat{f}\right]_Y q^*$.

Lemma 4.34. $\left[q_!\left(\hat{f}\right)_Y\right] \simeq q_!\left[\hat{f}\right]_Y q^*$

Proof. (of the lemma) The basic idea is the computation that,

$$q_!\left[\hat{f}\right]_Y q^* = \mathsf{colim}_{y\in Y}\left(\left[\hat{f}\right]_Y q^*\right)_y \simeq \mathsf{colim}_{y\in Y}\left[y^*\left(\hat{f}\right)_Y\right]y^*q^* \simeq \left[\mathsf{colim}_{y\in Y}\, y^*\left(\hat{f}\right)_Y\right] = \left[q_!\left(\hat{f}\right)_Y\right]$$

More precisely, there is an (essentially) commutative square

$$\begin{array}{ccc} \mathsf{Fun}\left(Y,\mathcal{S}_n^{n-1}\right) & \xrightarrow{\quad q_! \quad} & \mathcal{S}_n^{n-1} \\ {\scriptstyle [-]_*}\downarrow & & \downarrow{\scriptstyle [-]} \\ \mathsf{Fun}\left(Y,\mathsf{Fun}\left(\mathcal{C},\mathcal{C}\right)\right) & \xrightarrow[q_!]{} & \mathsf{Fun}\left(\mathcal{C},\mathcal{C}\right) \end{array}$$

such that the image of $\left(\hat{f}\right)_Y$ in $\mathsf{Fun}\left(Y,\mathsf{Fun}\left(\mathcal{C},\mathcal{C}\right)\right)$ is precisely $\left[\hat{f}\right]_Y q^*$. ☐

It light of the lemma, it suffices to show that $\hat{f}$ can be identified with the colimit of $\left(\hat{f}\right)_Y$ in $\mathcal{S}_n^{n-1}$. $\left(\hat{f}\right)_Y$ was obtained as the image under the straightening construction $\mathcal{S}_n{/}_Y \xrightarrow[\sim]{\mathsf{St}} \mathsf{Fun}\left(Y,\mathcal{S}_n\right)$, of the span of spaces over Y,

$$\begin{array}{ccccc} & & X & & \\ & {\scriptstyle f}\swarrow & & \diagdown\!\!\diagdown & \\ Y & & & & X \\ & \diagdown\!\!\diagdown & & \swarrow{\scriptstyle f} & \\ & & Y & & \end{array}$$

This can be alternatively described in terms of an infinity category of spans of spaces over Y. If we let $\left(\mathcal{S}_n/Y\right)^\dagger$ denote the weak CoWaldhausen structure on $\mathcal{S}_n/Y$ given by the $(n-1)$-truncated maps over Y, then we have the correspondence

Lemma 4.35. *Under the straightening construction* $\mathcal{S}_n/Y \xrightarrow[\sim]{\mathsf{St}} \mathsf{Fun}\,(Y, \mathcal{S}_n)$, $\left(\mathcal{S}_n/Y\right)^\dagger$ *corresponds to the weak CoWaldhausen structure* $\left(\mathcal{S}_n^Y\right)^\dagger$ *given by the morphisms that are object-wise* $(n-1)$*-truncated.*

Proof. (of the lemma) Consider a map over Y

$$\begin{array}{ccc} A & \xrightarrow{\quad f \quad} & B \\ & {}_{p}\searrow \quad \swarrow_{q} & \\ & Y & \end{array}$$

For every object $\{y\} \xrightarrow{y} Y$, the map on the fibers $A_y \to B_y$ can be identified with the component at y of the image of f under straightening. This fits into a composition of pullbacks

$$\begin{array}{ccccc} A_y & \xrightarrow{f_y} & B_y & \xrightarrow{q_y} & \{y\} \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow y \\ A & \xrightarrow[f]{} & B & \xrightarrow[q]{} & Y \end{array}$$

Consequently, if $A \to B$ is $(n-1)$-truncated, so is each $A_y \to B_y$. Further, the fiber of $A \to B$ over an object b of B is a fiber of $A_y \to B_y$, for $y = qb$. It follows that dually, if each $A_y \to B_y$ is $(n-1)$-truncated, $A \to B$ is as well. □

We have already seen in Lemma 2.10 that the colimit of a functor $Y \to \mathcal{S}_n$ can be identified with the total space of the Kan fibration over Y that it classifies. This is in fact an incarnation of a commutative diagram

$$\begin{array}{ccc} \mathcal{S}_n/Y & \xrightarrow{\quad \mathsf{St} \quad} & \mathsf{Fun}\,(Y, \mathcal{S}_n) \\ & {}_{\pi}\searrow \quad \swarrow_{\mathsf{colim}} & \\ & \mathcal{S}_n & \end{array}$$

This is also a commutative diagram of weak CoWaldhausen infinity categories, and by the lemma the top map is an equivalence still. Thus on passing to spans we have

$$\begin{array}{ccc} \mathsf{Span}\left(\mathcal{S}_n/Y, \left(\mathcal{S}_n/Y\right)^\dagger\right) & \xrightarrow{\quad \mathsf{St} \quad} & \mathsf{Fun}\,(Y, \mathcal{S}_n^{n-1}) \\ & {}_{\pi}\searrow \quad \swarrow_{\mathsf{colim}} & \\ & \mathcal{S}_n^{n-1} & \end{array}$$

where the map on the right is indeed the colimit map, as we have seen in Observation 2.28 that the inclusion $\mathcal{S}_n \to \mathcal{S}_n^{n-1}$ preserves and detects Y-indexed colimits.

Thus in order to compute the colimit of $\left(\widehat{f}\right)_Y$, it suffices to compute the image of

$$\begin{array}{ccccc} & & X & & \\ & \swarrow^{f} & & \searrow^{=} & \\ Y & & & & X \\ & \searrow^{=} & & \swarrow^{f} & \\ & & Y & & \end{array}$$

under the map forgetting the last map to Y, which produces exactly the span $\widehat{f}$ as desired. ☐

Notation 4.36. Consider an infinity category $\mathcal{C}$ satisfying the standing hypothesis (Hypothesis 4.3). The data of an action also includes the data of an "associativity" isomorphism, that we denote as $\mathfrak{m}$.

More explicitly, we denote for n-finite spaces X, Y the component

$$\mathfrak{m}_{X,Y}\colon [X \times Y] \xrightarrow{\sim} [X] \circ [Y]$$

of $\mathfrak{m}$ at (X, Y)[13], a family of equivalences functorial in X and Y.

We are now ready to prove our desired criterion. Recall that this was,

Proposition 4.37. *(cf.* [Har20, Propostion 3.17]*) Let $\mathcal{C}$ be an infinity category satisfying the standing hypothesis (Hypothesis 4.3). Then, $\mathcal{C}$ is n-semiadditive if and only if for every n-finite space X, the transformation*

$$[X] \circ [X] \xrightarrow[\sim]{\mathfrak{m}^{-1}_{X,X}} [X \times X] \xrightarrow{[\mathsf{Tr}_X]} 1_{\mathcal{C}}$$

exhibits $[X]$ as self dual in $\mathsf{Fun}(\mathcal{C}, \mathcal{C})$.

Proof. What we will do is identify this map with the trace form TrFm_p, where $p\colon X \to *$ is the map to the point. We will then be done by Proposition 4.11, the criterion of [HopLur13, Proposition 5.1.8] relating ambidexterity with the trace form being a self-duality evaluation map.

Lemma 4.38. *For an infinity category $\mathcal{C}$ satisfying the standing hypothesis (Hypothesis 4.3), and n-finite spaces X, Y, form the pullback square*

$$\begin{array}{ccc} X \times Y & \xrightarrow{\pi_2} & Y \\ \big\downarrow{\scriptstyle \pi_1} & \lrcorner \quad \sigma & \big\downarrow{\scriptstyle p_Y} \\ X & \xrightarrow[p_X]{} & * \end{array}$$

Denoting by p_Z for a space Z the map $Z \to *$, the transformation

$$[X \times Y] \xrightarrow[\sim]{\mathfrak{m}_{X,Y}} [X] \circ [Y]$$

can be identified with

$$[X \times Y] \simeq (p_{X \times Y})_! \, p^*_{X \times Y} \simeq (p_X)_! \, (\pi_X)_! \, \pi^*_Y p^*_Y \xrightarrow{(p_X)_! \mathsf{BC}[\sigma] p^*_Y} (p_X)_! \, p^*_X \, (p_Y)_! \, p^*_Y \simeq [X] \circ [Y]$$

[13] Recall that the monoidal structure on $\mathcal{S}_n^{n-1}$ acts as the Cartesian product level-wise.

Proof. (of the lemma) Upon writing $X \simeq \mathsf{colim}_X *$, $Y \simeq \mathsf{colim}_Y *$, using the compatibility of actions with n-finite colimits and the identification $[*] \simeq 1_{\mathcal{C}}$, one sees that both maps are equivalent to the "Fubini isomorphism"

$$\mathsf{colim}_{X\times Y} 1_{\mathcal{C}} \simeq \mathsf{colim}_X \, \mathsf{colim}_Y \, 1_{\mathcal{C}}$$

□

Returning to the main proof, recall from Definition 4.31 that ν_p was defined by forming the diagram

$$\begin{array}{ccc} X \times X & \xrightarrow{\pi_2} & X \\ \Big\downarrow{\scriptstyle \pi_1} & \sigma & \Big\downarrow{\scriptstyle p} \\ X & \xrightarrow[p]{} & * \end{array}$$

and taking the composition

$$\nu_p \colon p^* p_! \xrightarrow[\sim]{\mathsf{BC}[\sigma]^{-1}} \pi_{1!}\pi_2^* \xrightarrow{\pi_{1!}\left[\widehat{\delta}\right]_{X\times X}\pi_2^*} \pi_{1!}\delta_!\delta^*\pi_2^* \simeq 1_{\mathcal{C}_X} \circ 1_{\mathcal{C}_X} \simeq 1_{\mathcal{C}_X}$$

with δ being the diagonal of X. The composite $(p_!\nu_p p^*) \circ \mathfrak{m}_{X,X}$ can thus be identified with

$$p_!\pi_{1!}\left[\widehat{\delta}\right]_{X\times X}\pi_2^* p^* \simeq q_!\left[\widehat{\delta}\right]_{X\times X} q^*$$

where by q we denote $p_{X\times X} \colon X \times X \to *$.

But we already know from Proposition 4.33 that this last map is equivalent to the map induced directly by the dual span $\left[\widehat{\delta}\right] : [X \times X] \to [X]$. Thus

$$[p] \circ (p_!\nu_p p^*) \circ \mathfrak{m}_{X,X} \simeq [p] \circ \left[\widehat{\delta}\right] \simeq \left[p \circ \widehat{\delta}\right] \simeq [\mathsf{Tr}_X]$$

Consequently we may identify the transformation of the hypothesis, $[\mathsf{Tr}_X] \circ \mathfrak{m}_{X,X}^{-1}$ with $[p] \circ (p_!\nu_p p^*)$. We have proved in Proposition 4.19 that $[p] : p_!p^* \to 1_{\mathcal{C}}$ is a counit for the adjunction $p_! \dashv p^*$, and in Definition 4.31 that ν_p is homotopic to the ν_p^k of [HopLur13]. Thus $[p] \circ (p_!\nu_p p^*)$ and in turn $[\mathsf{Tr}_X] \circ \mathfrak{m}_{X,X}^{-1}$ is homotopic to the TrFm_X of [HopLur13], as desired. We may thus conclude the proposition. □

As remarked before when discussing Proposition 4.15, from the proposition we have the corollaries that infinity categories with n-finite colimits and a suitable action of $\mathcal{S}_n^n$ are n-semiadditive, and in particular that each $\mathcal{S}_n^m$ is m-semiadditive.

We conclude this section by quoting without proof another succinct criterion for semiadditivity.

Corollary 4.39. [Har20, Cor 3.18] *For an infinity category $\mathcal{C}$ satisfying the standing hypothesis (Hypothesis 4.3), $\mathcal{C}$ is n-semiadditive if and only if for every n-finite space X, the canonical colimit cocone of Construction 2.12 $*_X \xrightarrow{(i_x)} X_X$ induces a cone*

$$\left[\widehat{i_x}\right] : [X]_X \to [*]_X$$

which is a limit cone, establishing $[X] \simeq \mathsf{lim}_X 1_{\mathcal{C}}$ in $\mathsf{Fun}(\mathcal{C}, \mathcal{C})$.

5. The universal property of $\mathcal{S}_n^m$

The main result of [Har20] is the following universal property of $\mathcal{S}_n^m$. Recall from Notation 2.30 that for infinity categories $\mathcal{C}$ and $\mathcal{D}$ with n-finite colimits, we denote by $\mathsf{Fun}_{\kappa_n}(\mathcal{C}, \mathcal{D})$ the full subcategory of $\mathsf{Fun}(\mathcal{C}, \mathcal{D})$ spanned by the functors preserving n-finite colimits.

Theorem 5.1. *(cf.* [Har20, Theorem 4.1]*) Take integers $-2 \leqslant m \leqslant n$, and $\mathcal{C}$ an m-semiadditive infinity category with n-finite colimits. Then, evaluation at the point defines an equivalence*

$$\mathsf{ev}_*\colon \mathsf{Fun}_{\kappa_n}(\mathcal{S}_n^m, \mathcal{C}) \xrightarrow{\sim} \mathcal{C}$$

That is, $\mathcal{S}_n^m$ is the universal m-semiadditive infinity category admitting n-finite colimits generated by the point.

Remark 5.2. This universal property, and most of the rest of this section is due to [Har20]. The proof of the universal property itself is a clever yet somewhat technical use of particular Kan-extensions to demonstrate that the evaluation functor factors through several equivalences.

We will instead try to demonstrate how this universal property can be used to characterize higher semiadditivity, and to establish that the infinity category Cat_{κ_n} of n-semiadditive infinity categories and n-finite colimit preserving functors is itself n-semiadditive. Regardless, we will mostly follow Harpaz throughout, except that we chose to provide a slightly different proof of the fact that Cat_{κ_n} is n-semiadditive.

5.1. The criterion for higher semiadditivity.

Corollary 5.3. *Every m-semiadditive infinity category with m-finite colimits has a canonical $\mathcal{S}_m^m$-action preserving m-finite colimits variable-wise.*

Proof. (of the corollary) We transfer along the equivalence of the theorem the canonical pre-composition action of $\mathcal{S}_m^m$ on $\mathsf{Fun}_{\kappa_m}(\mathcal{S}_m^m, \mathcal{C})$ □

We have thus finally established (one implication from Corollary 4.17 of the last section and the other by Corollary 5.3 above) our characterization of n-semiadditivity as having an action of $\mathcal{S}_n^n$ compatible with n-finite colimits. We can promote this to an equivalence of categories, essentially identifying the appropriate notion of functor between the first ($\mathcal{S}_n^n$-module functors) with those of the second (functors preserving n-finite (co)limits).

Definition 5.4. For an integer $n \geqslant -2$, let $\mathsf{SAdd_n}$ be the full subcategory of Cat_{κ_n} spanned by the n-semiadditive infinity categories. It can be identified as the infinity category of n-semiadditive infinity categories and functors preserving n-finite (co)limits.

Notation 5.5. We denote by $\mathcal{U}\colon \mathsf{Mod}_{\mathcal{S}_n^n}(\mathsf{Cat}_{\kappa_n}) \to \mathsf{Cat}_{\kappa_n}$ the forgetful functor.

Observation 5.6. *By our characterization of n-semiadditivity, we see that $\mathcal{U}$ induces an essentially surjective $\mathsf{Mod}_{\mathcal{S}_n^n}(\mathsf{Cat}_{\kappa_n}) \xrightarrow{\mathcal{U}} \mathsf{SAdd_n}$.*

Our goal is to show that this map is an equivalence. It is essentially surjective, it only remains to show that it is fully-faithful. As it is the right adjoint in a free-forgetful adjunction, we need only equivalently show that the counit of this adjunction is invertible.

Notation 5.7. Let $\mathcal{F} := \mathcal{S}_n^n \otimes_{\mathsf{Cat}_{\kappa_n}} (-) : \mathsf{Cat}_{\kappa_n} \to \mathsf{Mod}_{\mathcal{S}_n^n}(\mathsf{Cat}_{\kappa_n})$ denote the left adjoint to $\mathcal{U}$.

Denote by η and ε respectively, the unit and counit of this adjunction.

Proposition 5.8. *The counit ε is invertible, that is, the right adjoint $\mathcal{U}$ is fully-faithful.*

Proof. We must show for an arbitrary $\mathcal{S}_n^n$-module $\mathcal{C}$, that the counit map

$$\varepsilon_{\mathcal{C}} \colon \mathcal{S}_n^n \otimes_{\mathsf{Cat}_{\kappa_n}} \mathcal{U}\mathcal{C} \to \mathcal{C}$$

is invertible in $\mathsf{Mod}_{\mathcal{S}_n^n}(\mathsf{Cat}_{\kappa_n})$. As $\mathcal{U}$ is conservative, it suffices to check that the induced functor $\mathcal{U}\varepsilon_{\mathcal{C}}$ of underlying infinity categories is invertible.

Lemma 5.9. *The unit map at an infinity category $\mathcal{C}$,*

$$\eta_{\mathcal{C}} \colon \mathcal{C} \to \mathcal{U}\left(\mathcal{S}_n^n \otimes_{\mathsf{Cat}_{\kappa_n}} \mathcal{C}\right)$$

is invertible if and only if $\mathcal{C}$ is n-semiadditive.

Proof. (of the lemma) First, if the unit map is an equivalence, then $\mathcal{C}$ is n-semiadditive as $\mathcal{U}\left(\mathcal{S}_n^n \otimes_{\mathsf{Cat}_{\kappa_n}} \mathcal{C}\right)$ necessarily is. For the converse, let $\mathcal{C}$ be n-semiadditive.

For an arbitrary $\mathcal{D} \in \mathsf{Cat}_{\kappa_n}$, restriction along $\eta_{\mathcal{C}}$ defines a

$$\mathsf{Fun}_{\kappa_n}\left(\mathcal{U}\left(\mathcal{S}_n^n \otimes_{\mathsf{Cat}_{\kappa_n}} \mathcal{C}\right), \mathcal{D}\right) \xrightarrow{\eta_{\mathcal{C}}^*} \mathsf{Fun}_{\kappa_n}(\mathcal{C}, \mathcal{D})$$

In light of the adjunction equivalence $\mathsf{Fun}_{\kappa_n}\left(\mathcal{U}\left(\mathcal{S}_n^n \otimes_{\mathsf{Cat}_{\kappa_n}} \mathcal{C}\right), \mathcal{D}\right) \simeq \mathsf{Fun}_{\kappa_n}(\mathcal{S}_n^n, \mathsf{Fun}_{\kappa_n}(\mathcal{C}, \mathcal{D}))$, this restriction can be identified with the evaluation

$$\mathsf{ev}_* \colon \mathsf{Fun}_{\kappa_n}(\mathcal{S}_n^n, \mathsf{Fun}_{\kappa_n}(\mathcal{C}, \mathcal{D})) \simeq \mathsf{Fun}_{\kappa_n}\left(\mathcal{U}\left(\mathcal{S}_n^n \otimes_{\mathsf{Cat}_{\kappa_n}} \mathcal{C}\right), \mathcal{D}\right) \xrightarrow{\eta_{\mathcal{C}}^*} \mathsf{Fun}_{\kappa_n}(\mathcal{C}, \mathcal{D})$$

which the universal property of $\mathcal{S}_n^n$ guarantees is an equivalence. By the Yoneda lemma applied to Cat_{κ_n}, the restriction being an equivalence for arbitrary $\mathcal{D}$ implies that $\eta_{\mathcal{C}}$ is an equivalence, proving the converse. □

$\mathcal{U}\varepsilon_{\mathcal{C}}$ is (by the triangle identity) a retract of $\eta_{\mathcal{U}\mathcal{C}}$. As $\mathcal{U}\mathcal{C}$ is n-semiadditive, the lemma tells us that $\eta_{\mathcal{U}\mathcal{C}}$ is invertible, and hence its retract $\mathcal{U}\varepsilon_{\mathcal{C}}$ is as well. By conservativity, the counit is invertible as desired. □

Corollary 5.10. $\mathcal{U} \colon \mathsf{Mod}_{\mathcal{S}_n^n}(\mathsf{Cat}_{\kappa_n}) \to \mathsf{Cat}_{\kappa_n}$ *defines an equivalence*

$$\mathsf{Mod}_{\mathcal{S}_n^n}(\mathsf{Cat}_{\kappa_n}) \xrightarrow{\sim} \mathsf{SAdd_n}$$

Corollary 5.11. *The inclusion $\mathsf{SAdd_n} \subseteq \mathsf{Cat}_{\kappa_n}$ has both a left and a right adjoint*[14]*, given on objects as $\mathcal{C} \mapsto \mathcal{S}_n^n \otimes_{\mathsf{Cat}_{\kappa_n}} \mathcal{C}$ and $\mathcal{C} \mapsto \mathsf{Fun}_{\kappa_n}(\mathcal{S}_n^n, \mathcal{C})$ respectively.*

5.2. **Example: Cat_{κ_n} is n-semiadditive.** As a final demonstration of the utility of this universal property, we show how it can be used to prove that Cat_{κ_n} is n-semiadditive. This is also demonstrated in [Har20], but we provide a slightly different proof.

Proposition 5.12. *(cf.* [Har20, Proposition 5.26]*) Consider an integer $n \geqslant -2$. The infinity category Cat_{κ_n} is n-semiadditive.*

Proof. We will prove by induction on an integer m, that Cat_{κ_n} is m-semiadditive for every $-2 \leqslant m \leqslant n$. The case for $m = -2$ is known to hold a-priori, so we reduce to having to show that for $m \geqslant -1$, Cat_{κ_n} is m-semiadditive under the assumption that it is $(m-1)$-semiadditive.

Our first observation is that since an infinity category is k-semiadditive if and only if its opposite is, it suffices to do the same for $\mathsf{Cat}_{\kappa_n}^{\mathsf{op}}$. Cat_{κ_n} is complete, with limits computed as in Cat_∞. Hence $\mathsf{Cat}_{\kappa_n}^{\mathsf{op}}$ is cocomplete, and in particular has a canonical action of $\mathcal{S}$. Explicitly, this is computed as $[X]\mathcal{C} \simeq \mathsf{colim}_X\, \mathcal{C}$ (in $\mathsf{Cat}_{\kappa_n}^{\mathsf{op}}$). As in Cat_{κ_n} we can identify $\mathsf{Fun}(X, \mathcal{C}) \simeq \mathsf{lim}_X\, \mathcal{C}$[15], we ultimately have $[X]\mathcal{C} \simeq \mathcal{C}^X$. Further, the action of a $u \colon X \to Y$ in $\mathcal{S}$ on $\mathcal{C}$ is the map between the respective colimits given by the action of u on the indices. In terms of the identifications of the form $[X]\mathcal{C} \simeq \mathcal{C}^X$,

[14]Transferred from the left and right adjoints to the forgetful functor $\mathcal{U} \colon \mathsf{Mod}_{\mathcal{S}_n^n}(\mathsf{Cat}_{\kappa_n}) \to \mathsf{Cat}_{\kappa_n}$.

[15]The cone maps being given by evaluation.

the action $[u]\mathcal{C}\colon [X]\mathcal{C} \to [Y]\mathcal{C}$ is the map in $\mathsf{Cat}_{\kappa_n}^{\mathsf{op}}$ corresponding to the pullback map $u^*\colon \mathcal{C}^Y \to \mathcal{C}^X$ in Cat_{κ_n}.

In any case, we now work in the context of an $(m-1)$-semiadditive cocomplete infinity category Cat_{κ_n}. Thus Theorem 5.1, Harpaz's universal property of $\mathcal{S}_m^{m-1}$ (cf. [Har20, Theorem 4.1]) asserts that evaluation at the point defines an equivalence

$$\mathsf{ev}_*\colon\ \mathsf{Fun}_{\kappa_m}\left(\mathcal{S}_m^{m-1}, \mathsf{Cat}_{\kappa_n}^{\mathsf{op}}\right) \xrightarrow{\sim} \mathsf{Cat}_{\kappa_n}^{\mathsf{op}}$$

and thus $\mathsf{Cat}_{\kappa_n}^{\mathsf{op}}$ inherits an action of $\mathcal{S}_m^{m-1}$ compatible with m-finite colimits variable-wise (by transferring the precomposition action as before). Or put differently, $\mathsf{Cat}_{\kappa_n}^{\mathsf{op}}$ satisfies the *standing hypothesis (Hypothesis 4.3)* of the previous section.

Our strategy now will be to appeal to Proposition 4.37 of the previous section, which in our case will guarantee that $\mathsf{Cat}_{\kappa_n}^{\mathsf{op}}$ is m-semiadditive as long as we can show that for every $X \in \mathcal{S}_m^{m-1}$, $[\mathsf{Tr}_X]$ exhibits $[X]$ as self dual (in $\mathsf{Fun}_{\kappa_m}\left(\mathsf{Cat}_{\kappa_n}^{\mathsf{op}}, \mathsf{Cat}_{\kappa_n}^{\mathsf{op}}\right)$). For this, we will describe the action of a span $Y \xleftarrow{q} Z \xrightarrow{p} X$.

The first observation is that both the action of $\mathcal{S}$ and $\mathcal{S}_m^{m-1}$ on $\mathsf{Cat}_{\kappa_n}^{\mathsf{op}}$ are induced by transferring precomposition actions along equivalences given by evaluating at the point. In particular they fit into a diagram,

$$\begin{array}{ccc} \mathsf{Fun}_{\kappa_m}\left(\mathcal{S}_m^{m-1}, \mathsf{Cat}_{\kappa_m}^{\mathsf{op}}\right) & \xrightarrow{\mathsf{ev}_*} & \mathsf{Cat}_{\kappa_m}^{\mathsf{op}} \\ \downarrow & \nearrow_{\mathsf{ev}_*} & \uparrow{\scriptstyle \mathsf{ev}_*} \\ \mathsf{Fun}_{\kappa_m}\left(\mathcal{S}_m, \mathsf{Cat}_{\kappa_m}^{\mathsf{op}}\right) & \longleftarrow & \mathsf{Fun}_!\left(\mathcal{S}, \mathsf{Cat}_{\kappa_m}^{\mathsf{op}}\right) \end{array}$$

and thus agree on the common restrictions to $\mathcal{S}_m$ (justifying our use of the notation $[-]$ for both).

Next, we show that $[X]$ is indeed self-dual (although we will not do so by showing directly that $[\mathsf{Tr}_X]$ is a coevaluation). The fact each $\mathcal{C} \in \mathsf{Cat}_{\kappa_m}$ has m-finite colimits is a guarantee of sufficiently many colimits to let us assign to every span $Y \xleftarrow{q} Z \xrightarrow{p} X$ of m-finite spaces the map

$$\tau_{q,p}\colon \mathcal{C}^X \xrightarrow{p^*} \mathcal{C}^Z \xrightarrow{q_!} \mathcal{C}^Y$$

In light of the Beck-Chevalley property, the aforementioned τ-construction is still compatible with composition (up to equivalence). It is similarly compatible with the monoidal structure (in light of the identifications $\mathcal{C}^{X\times X} \simeq \left(\mathcal{C}^X\right)^X$), and thus preserves dualizable objects and is also compatible with duals. Thus the map $\tau_{\delta_X,p}$ corresponding to Tr_X is a coevaluation establishing $[X]$ as self dual for every X. The only thing remaining is to identify $\tau_{\delta_X,p}$ and $[\mathsf{Tr}_X]$.

Lemma 5.13. *For an infinity category $\mathcal{C}$, the action on $\mathcal{C}$ of the dual span $\widehat{f}$ to an $(m-1)$-truncated $f\colon X \to Y$ is a morphism in $\mathsf{Cat}_{\kappa_m}^{\mathsf{op}}$ corresponding to $\mathcal{C}^X \xrightarrow{f_!} \mathcal{C}^Y$ given by left Kan-extension along f.*

Proof. We first show this for a special case, that of a morphism $f\colon X \to Y$ between $(m-1)$-finite spaces X, Y. In $\mathcal{S}_m^{m-1}$, the $(m-1)$-finite spaces are dualizable, as both the trace and its dual span are well defined. Thus we can speak of the monoidal dual of f in $\mathcal{S}_m^{m-1}$, and this is indeed seen to be $\widehat{f}$. Thus, in particular, $\widehat{f}$ acts as the dual of the action of f. The strategy will be to identify the morphism in Cat_{κ_m} corresponding to the dual of $[f]$ with $f_!$. As we know already that $[f]$ is the morphism in $\mathsf{Cat}_{\kappa_m}^{\mathsf{op}}$ corresponding to f^*, it will follow that the action of $\widehat{f}$ can be identified with the morphism in $\mathsf{Cat}_{\kappa_m}^{\mathsf{op}}$ corresponding to $f_!$.

As we have just seen that for every object X there is an evaluation-coevaluation self duality pair $\tau_{p,\delta_X}, \tau_{\delta_X,p}$ (where p is the map $X \to *$). The dual of $[f]$ as the morphism acting on a $\mathcal{C} \in \mathsf{Cat}_{\kappa_m}^{\mathsf{op}}$ is the morphism in Cat_{κ_m}

$$[Y]\,\mathcal{C} \xrightarrow{[Y]\tau_{p,\delta_X}} [Y]\,[X]^2\,\mathcal{C} \xrightarrow{[Y][f][X]} [Y]^2\,[X]\,\mathcal{C} \xrightarrow{\tau_{\delta_X,p}[X]} [X]\,\mathcal{C}$$

this corresponds to the morphism in Cat_{κ_m} given by

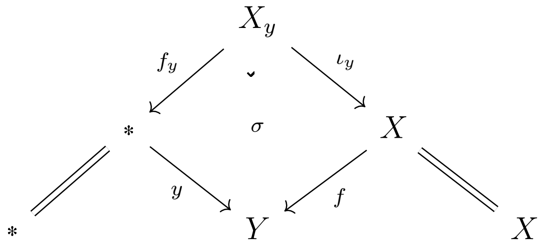
$$\mathcal{C}^X \xrightarrow{\pi_X^*} \mathcal{C}^{X\times Y} \xrightarrow{(1_Y\times\delta_Y)_!} \mathcal{C}^{X\times Y\times Y} \xrightarrow{(1_X\times f\times 1_Y)^*} \mathcal{C}^{X\times X\times Y} \xrightarrow{(\delta_X\times 1_Y)^*} \mathcal{C}^{X\times Y} \xrightarrow{(p_!)_*} \mathcal{C}^Y$$

which by the colimit formula for left Kan-extensions can be seen to precisely compute the left Kan-extension (point-wise). This establishes the special case.

Now, for a general $(m-1)$-truncated $f\colon X \to Y$ in $\mathcal{S}_m$ and a $\mathcal{C} \in \mathsf{Cat}_{\kappa_m}$, we will check that $\left[\widehat{f}\right]\mathcal{C}$ corresponds to a morphism $\mathcal{C}^X \to \mathcal{C}^Y$ in Cat_{κ_m} computing (point-wise) left Kan-extension. For an object $y \in Y$, consider the fiber

$$\begin{array}{ccc} X_y & \xrightarrow{\iota_y} & X \\ \big\downarrow{\scriptstyle f_y} & \lrcorner\ \sigma & \big\downarrow{\scriptstyle f} \\ \{y\} & \xrightarrow[y]{} & Y \end{array}$$

The square σ fits into a diagram



from which we see that the span $* \xleftarrow{f_y} X_y \xrightarrow{\iota_y} X$ is the composite $\widehat{f}\circ y$. Thus in particular $\left[\widehat{f}\right]\circ[y]$ is the action of this span.

This acts on $\mathcal{C}$ as $[*]\,\mathcal{C} \xrightarrow{[y]\mathcal{C}} [Y]\,\mathcal{C} \xrightarrow{[\widehat{f}]\mathcal{C}} [X]\,\mathcal{C}$. Denoting by λ the morphism in Cat_{κ_m} corresponding to the action $\left[\widehat{f}\right]\mathcal{C}$ in $\mathsf{Cat}_{\kappa_m}^{\mathsf{op}}$, the span thus acts as

$$\mathcal{C}^X \xrightarrow{\lambda} \mathcal{C}^Y \xrightarrow{y^*=\mathsf{ev}_y} \mathcal{C}$$

and computes the action of λ at $y \in Y$. However the span $* \xleftarrow{f_y} X_y \xrightarrow{\iota_y} X$ is also the composition $\iota_y \circ \widehat{f}_y$ and thus acts as $[\iota_y]\circ\left[\widehat{f}_y\right]\mathcal{C}$ in $\mathsf{Cat}_{\kappa_m}^{\mathsf{op}}$. We know that ι_y acts as ι_y^* and by the special case of maps between $(m-1)$-finite spaces we know that f_y acts as $(f_y)_!$. Thus the action is the map in $\mathsf{Cat}_{\kappa_m}^{\mathsf{op}}$ corresponding to the morphism $\mathcal{C}^X \xrightarrow{\iota_y^*} \mathcal{C}^{X_y} \xrightarrow{(f_y)_!} \mathcal{C}$ in Cat_{κ_m}. But as we have the Beck-Chevalley identification $(f_y)_!\,\iota_y^* \simeq y^* f_!$, we thus have constructed an identification $y^*\lambda \simeq y^* f_!$. Therefore, λ acts as the (point-wise) left Kan-extension, as desired. $\square$

Consequently, by decomposing the span $Y \xleftarrow{q} Z \xrightarrow{p} X$ as the composition of p and the dual span to q, we see that it acts on an infinity category $\mathcal{C}$ as precisely the map $\tau_{q,p}\colon\ [Y]\,\mathcal{C} \to [X]\,\mathcal{C}$ in $\mathsf{Cat}_{\kappa_m}^{\mathsf{op}}$. Thus, denoting by $p\colon X \to *$ the map to the point, we can identify $[\mathsf{Tr}_X]$ and $\tau_{\delta_X,p}$. This lets us conclude the inductive step by the trace criterion, and therefore by induction the proposition. $\square$

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Einverständniserklärung

Ich habe die Arbeit selbstständig verfasst, keine anderern als die angegebenen Quellen und Hilfsmittel benutzt und bisher keiner anderen Prüfungsbehörde vorgelegt.
Außerdem bestätige ich hiermit, dass die vorgelegten Druckexemplare und die vorgelegte elektronische Version der Arbeit identisch sind und dass ich von den in §26 Abs. 6 vorgesehenen Rechtsfolgen Kenntnis habe.

Signed: ......Areeb..............
Areeb Shah Mohammed
Regensburg, den 16.NOV.2022