HOMOLOGY, COHOMOLOGY AND PRODUCTS

SEMINAR: *INTRODUCTION TO STABLE HOMOTOPY THEORY*
AREEB SHAH-MOHAMMED

Contents

1. Homology and Cohomology 1
2. Products 6
References 11

Remark 0.1. We will primarily follow [Adams, Sections 6,9] from which we take most of the first section of this report and much of the second. We also cite [Switzer, in particular Chapter 13 on products] for the definitions of (commutative) ring spectra and the statements related to them, where in addition to the proofs we reproduce one can also find proofs for many of the claims stated without them.

1. Homology and Cohomology

Remark 1.1. We will assume all spectra to be CW-spectra. Similarly, all pairs of spectra will be pairs (X, A) for X a CW-spectrum and A a CW-subspectrum.

Notation 1.2. We denote by $\mathbb{S}$ the Sphere spectrum $\Sigma^\infty S^0$.

Notations 1.3. We denote by $\mathsf{hCW\text{-}Spec}$ the homotopy category of CW-spectra. We denote by GrAb the category of $\mathbb{Z}$-graded abelian groups and degree preserving group homomorphisms.

Definition 1.4. The functor $\wedge\colon \mathsf{hCW\text{-}Spec} \times \mathsf{hCW\text{-}Spec} \to \mathsf{hCW\text{-}Spec}$ defines on composition with the "hom functor" $[\mathbb{S}, -]\colon \mathsf{hCW\text{-}Spec} \to \mathsf{GrAb}$ a functor

$$[\mathbb{S}, -] \circ \wedge\colon \mathsf{hCW\text{-}Spec} \times \mathsf{hCW\text{-}Spec} \to \mathsf{GrAb}$$

which assigns functorially to each E a functor $E_*\colon \mathsf{hCW\text{-}Spec} \to \mathsf{GrAb}$ such that

$$E_n(X) := [\mathbb{S}, E \wedge X]_n$$

which we call E-homology. Equivalently, we may define it as:

$$E_n(X) = [\mathbb{S}^n, E \wedge X]$$

where $\mathbb{S}^n = \Sigma^n \mathbb{S}$ (for instance, to avoid dealing explicitly with spectrum maps of non-zero degree).

The action of E_* on a morphism $f\colon X \to Y$ is given by post-composition by $1_E \wedge f$ and is denoted $f_* := E_*(f)\colon E_*(X) \to E_*(Y)$.

Definition 1.5. For each spectrum E we also assign functorially in E a functor $E^*\colon \mathsf{hCW\text{-}Spec}^{op} \to \mathsf{GrAb}$, the E-cohomology such that

$$E^n(X) := [X, E]_{-n}$$

Equivalently, we may define it as

$$E^n(X) = [X, \Sigma^n E]$$

The action of E^* on a morphism $f\colon X \to Y$ is by pre-composition and is denoted by $f^* := E^*(f)\colon E^*(Y) \to E^*(X)$.

Notation 1.6. The constructions of E-homology and E-cohomology are also functorial in the spectrum E, and in particular a morphism $f\colon E \to F$ induces $f_*\colon E_*(-) \to F_*(-)$ (resp. $f_*\colon E^*(-) \to F^*(-)$) by post-composition by $f \wedge 1$ (resp. f).

Proposition 1.7. *A cofibre sequence $X \to Y \to Z$ of spectra induces for each spectrum E exact sequences*

$$E_n(X) \to E_n(Y) \to E_n(Z)$$
$$E^n(Z) \to E^n(Y) \to E^n(X)$$

Similarly, a cofibre sequence $E \to F \to G$ of spectra induces for each spectrum X exact sequences

$$E_n(X) \to F_n(X) \to G_n(X)$$
$$E^n(X) \to F^n(X) \to G^n(X)$$

Proof. This is ultimately a translation of the result proved earlier in the seminar that the hom-functors $([-, E], [X, -])$ take cofibre sequences to exact sequences.

The only other observation required (for the case of homology) is that smashing with a fixed spectrum preserves cofibre sequences. This follows as the proposition reduces to checking that this holds for the "naive" smash product $\wedge_{BC}$ with B, C infinite and the sequence $A \to X \to X \cup CA$ (for A a subspectrum of X). □

Proposition 1.8. *There are natural isomorphisms*

$$E_n(X) \cong E_{n+1}(\mathbb{S}^1 \wedge X)$$
$$E^n(X) \cong E^{n+1}(\mathbb{S}^1 \wedge X)$$

Observation 1.9. *For a spectrum E, $E_n(\mathbb{S}) \cong E^{-n}(\mathbb{S}) = \pi_n(\mathbb{S})$*

Definition 1.10. For a spectrum E, define functors $\tilde{E}_*\colon \mathsf{hCW}_* \to \mathsf{GrAb}$ and $\tilde{E}^*\colon \mathsf{hCW}_*^{op} \to \mathsf{GrAb}$ obtained by restricting E_* and E^* along the suspension spectrum functor $\Sigma^\infty\colon \mathsf{hCW}_* \to \mathsf{hCW\text{-}Spec}$. That is,

$$\tilde{E}_*(X) := E_*(\Sigma^\infty X)$$
$$\tilde{E}^*(X) := E^*(\Sigma^\infty X)$$

Observation 1.11. *As the functor Σ^∞ commutes with suspension and preserves cofibre sequences, the preceding two propositions imply that the functors $\tilde{E}_*, \tilde{E}^*$ satisfy the long exact sequence and suspension axioms for a reduced (co)homology theory in the sense of Eilenberg-Steenrod.*

Proposition 1.12. *For a spectrum E, $\tilde{E}_*$ is a reduced homology theory and $\tilde{E}^*$ is a reduced cohomology theory on based CW-complexes.*

Conversely, any reduced (co)homology theory on based CW-complexes derives from a spectrum in this manner.

Remarks 1.13. To see that the functors so defined are indeed (co)homology theories, in light of the above observation we need only show that $\tilde{E}_*, \tilde{E}^*$ satisfy the wedge axiom. For cohomology, this is direct from the definition in terms of hom-functors as we have shown that the wedge is the coproduct in hCW-Spec. Conversely, by Brown representability one shows that every reduced cohomology theory derives from a spectrum, as in Talk 2. One concludes the proposition for the case of cohomology.

The case for homology is more subtle. We have already seen that for finite wedges, it can be deduced from the cofibre sequence $X \to X \vee Y \to Y$. One shows that for a spectrum E the functor $\tilde{E}_*$ satisfies for a CW-complex X

$$\tilde{E}_*(X) = \mathrm{colim}_{Y \subseteq X, Y \text{ finite}} \tilde{E}_*(Y)$$

and one deduces the full wedge axiom for homology. For the converse, Adams (cf. [Adams, Section 6, Page 200]) sketches a proof that any reduced homology theory on spaces satisfying also the above colimit condition arises from a spectrum in this manner. It can be shown ([Switzer, Remark 1 immediately following Cor 14.36]) that for any reduced homology theory in the sense of Eilenberg-Steenrod this colimit condition holds (using only the standard wedge axiom). Thus we see that the converse holds for homology as well.

Examples 1.14. Some (co)homology theories determined by spectra of interest are:

- If HA is the Eilenberg-Maclane spectrum associated to an abelian group A (recall $\pi_0(\mathsf{HA}) = A$ and $\pi_n(\mathsf{HA}) = 0$ otherwise) then the (co)homology theory so defined is ordinary reduced (co)homology with coefficients in A.
- The sphere spectrum $\mathbb{S}$ defines a homology theory that recovers the stable homotopy groups, and the cohomology theory it defines is stable cohomotopy.
- The spectrum KU which arises in complex K-theory defines a cohomology theory which agrees on finite CW-complexes with the definition of K-theory as the isomorphism classes of complex vector bundles over the space. The determination of the coefficient groups is precisely determined by (one version of) the (complex) Bott periodicity theorem. There are also variations on this theme such as K-theory with coefficients, connective K-theory, etc.
- The spectrum KO which arises in real K-theory plays an analogous role. The cohomology theory now determines isomorphism classes of real vector bundles on finite complexes, and the real Bott periodicity theorem describes its coefficient groups. Again there are connective variants with coefficients.
- Many Thom spectra determine bordism and cobordism theories. For instance MO, MSO and MU determine unoriented, oriented and complex (co)bordism.

Theorem 1.15. *(Whitehead) The equivalence $E \wedge X \xrightarrow{\sim} X \wedge E$ induces an isomorphism $E_*(X) \cong X_*(E)$.*

Corollary 1.16. *For abelian groups A and B, $\mathsf{HA}_*(\mathsf{HB}) \cong \mathsf{HB}_*(\mathsf{HA})$*

Definition 1.17. For an abelian group A, a Moore spectrum for A (alternatively, of type A) is a spectrum M such that

$$\begin{aligned} \pi_r(M) &= 0 && r < 0 \\ \pi_0(M) &= (\mathsf{H}\mathbb{Z})_0(M) = A \\ (\mathsf{H}\mathbb{Z})_r(M) &= 0 && r > 0 \end{aligned}$$

Remark 1.18. There exists a Moore spectrum for any abelian group A, indeed A has a free resolution

$$0 \to \mathbb{Z}^{\oplus B} \to \mathbb{Z}^{\oplus C} \to A \to 0$$

then the cofibre of the map $\bigvee_B \mathbb{S} \to \bigvee_C \mathbb{S}$ that induces $\mathbb{Z}^{\oplus B} \to \mathbb{Z}^{\oplus C}$ can be shown to define a Moore spectrum for A.

Definition 1.19. For a spectrum E, we define the corresponding "Spectrum with coefficients in A" to be $EA := E \wedge M$ where M is a Moore spectrum for A.

Remark 1.20. Much as in classical (co)homology theory on spaces there are several exact sequences associated to a change of coefficients. For instance, there is a sequence of the form

$$0 \to E_n(X) \otimes A \to (EA)_n(X) \to \mathrm{Tor}_1^{\mathbb{Z}}(E_{n-1}(X), A) \to 0$$

Classically given a reduced (co)homology theory E on based CW-complexes, one defines a (co)homology theory on CW-pairs by defining it on the pair (X, A) to be the value of E at the homotopy cofibre $X \cup CA \simeq X/A$, the construction of which is functorial in maps of pairs. The same can be done for spectra.

Definition 1.21. Given a spectrum E and a pair of spectra (X, A) (so A is a closed CW-subspectrum of X) we define the relative (co)homology as

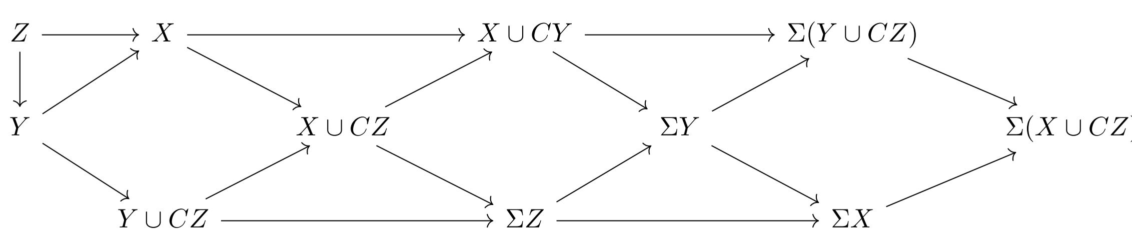
$$\begin{aligned} E_*(X, A) :=& E_*(X \cup CA) \cong E_*(X/A) \\ E^*(X, A) :=& E^*(X \cup CA) \cong E^*(X/A) \end{aligned}$$

The properties we have just shown for the functors E_* and E^* all translate into properties of the functors on pairs so defined. Further as Σ^∞ preserves cofibre sequences, the functors $E_*(-, \bullet), E^*(-, \bullet)$ restrict along it to give (co)homology theories on pairs of CW-complexes as expected.

One useful tool in computing these groups for spaces is the LES associated to a triple, i.e two nested subcomplex inclusions $Z \subseteq Y \subseteq X$. It then takes the form (for homology, and dually for cohomology)

$$\cdots \to E_n(Y, Z) \to E_n(X, Z) \to E_n(X, Y) \to E_{n-1}(Y, Z) \to \dots$$

One proves this for based spaces by constructing a "sine-wave" of cofibre sequences as below (commuting up to homotopy)



and taking the LES induced by the constructed cofibre sequence

$$Y \cup CZ \to X \cup CZ \to X \cup CY$$

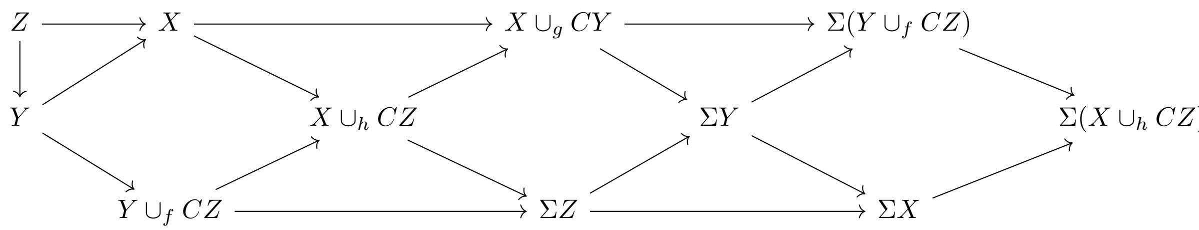
Now the same diagram can be constructed for spectra (and now commutes on the nose in the homotopy category), so we get the same cofibre sequence but now of spectra, and can take the associated LES.

We recall that this is done by extending the sequence, for instance to the right as

$$Y \cup CZ \to X \cup CZ \to X \cup CY \to \Sigma(Y \cup CZ) \to \dots$$

Then we know that applying E_n gives us an exact sequence, and in light of the suspension isomorphism $E_n(\Sigma(Y \cup CZ)) \cong E_{n-1}(Y \cup CZ)$ we get the desired sequence.

Remark 1.22. More generally, given $f\colon Z \to Y, g\colon Y \to X$ and setting $h := g \circ f$ one can form an analogous diagram of cofibre sequences



This is precisely the statement of Verdier's axiom. Indeed, one can show that hCW-Spec is triangulated when the cofibre sequences are taken to be the distinguished triangles.

Another tool used in computation is the Mayer-Vietoris sequence. Again, it will take the form of a cofibre sequence.

Proposition 1.23. *If a CW-spectrum is X is the union of two closed CW-subspectra U, V then there is a cofibre sequence*

$$U \cap V \xrightarrow{(i_U, -i_V)} U \vee V \xrightarrow{<i_U, i_V>} U \cup V = X$$

which can be continued by the translated cofibre sequence

$$U \vee V \xrightarrow{<i_U, i_V>} U \cup V = X \to \Sigma(U \cap V)$$

where the map $U \cup V \to \Sigma(U \cap V)$ can be described as the composite

$$U \cap V \to {}^{U \cap V}\!/_{V} \cong {}^{U}\!/_{U \cap V} \to \Sigma(U \cap V)$$

or alternatively as the additive inverse of the composite

$$U \cap V \to {}^{U \cap V}\!/_{U} \cong {}^{V}\!/_{U \cap V} \to \Sigma(U \cap V)$$

Corollary 1.24. *Given a spectrum E and a decomposition $X = U \cup V$ as in the proposition, continuing the cofibre sequence as usual and taking (co)homology gives long exact sequences*

$$\dots \to E_n(U \cap V) \to E_n(U \vee V) \cong E_n(U) \oplus E_n(V) \to E_n(U \cup V) \to E_{n-1}(U \cap V) \to \dots$$

$$\dots \to E^{n-1}(U \cap V) \to E^n(U \cup V) \to E^n(U \vee V) \cong E^n(U) \oplus E^n(V) \to E^n(U \cap V) \to \dots$$

2. Products

Classically, the Eilenberg-Zilber theorem establishes (inverse) chain equivalences

$$C_*(X) \otimes C_*(Y) \xrightarrow{\mu} C_*(X \times Y) \xrightarrow{\Delta} C_*(X) \otimes C_*(Y)$$

which induce by pre and post composition the "external products"

$$\mu_*\colon H_*(X) \otimes H_*(Y) \to H_*(X \times Y)$$
$$\Delta^*\colon H^*(X) \otimes H^*(Y) \to H^*(X \times Y)$$

One can also consider μ as encoding a $C_*(Y)$-parametrised family of maps $C_*(X) \to C_*(X \times Y) \leftrightarrow C^*(X \times Y) \to C^*(X)$, precisely a $C^*(X \times Y) \otimes C_*(Y) \to C^*(X)$ encoding a "slant-product"

$$H^*(X \times Y) \otimes H_*(Y) \to H^*(X)$$

Analogously, there is another slant product

$$H^*(X) \otimes H_*(X \times Y) \to H_*(Y)$$

induced by a similar map $C^*(X) \otimes C_*(X \times Y) \to C_*(Y)$.

We will perform analogous constructions for spectra. In fact, we will derive "products" (of the form $E_*(X) \otimes F_*(Y) \to G_*(X \wedge Y)$ for instance) given any map $E \wedge F \to G$ of spectra. Our strategy will be first to define products in the special case that this map is the identity (so $E_*(X) \otimes F_*(Y) \to (E \wedge F)_*(X \wedge Y)$ if we stick with the previous example). We will then use the functoriality of our (co)homology construction to get the full range of products.

Definitions 2.1. For spectra E, F, X, Y we have the following four products

- The exterior product in cohomology is a map
$$\bar{\wedge}\colon E^p(X) \otimes F^q(Y) \to (E \wedge F)^{p+q}(X \wedge Y)$$
$$f \otimes g \mapsto f \wedge g$$
for $X \xrightarrow{f} E \in E^p(X), Y \xrightarrow{g} F \in F^q(Y)$
- The exterior product in homology is a map
$$\underline{\wedge}\colon E_p(X) \otimes F_q(Y) \to (E \wedge F)_{p+q}(X \wedge Y)$$
$$f \otimes g \mapsto \mathbb{S} \cong \mathbb{S} \wedge \mathbb{S} \xrightarrow{f \wedge g} E \wedge X \wedge F \wedge G \cong E \wedge F \wedge X \wedge Y$$
for $\mathbb{S} \xrightarrow{f} E \wedge X \in E_p(X), \mathbb{S} \xrightarrow{g} F \wedge Y \in F_q(Y)$
- The first slant product is a map
$$/\colon E^p(X \wedge Y) \otimes F_q(Y) \to (E \wedge F)^{p-q}(X)$$
$$f \otimes g \mapsto X \xrightarrow{1 \wedge g} X \wedge F \wedge Y \cong X \wedge Y \wedge F \xrightarrow{f \wedge 1} E \wedge F$$
for $X \wedge Y \xrightarrow{f} E \in E^p(X \wedge Y), \mathbb{S} \xrightarrow{g} F \wedge Y \in F_q(Y)$
- The second slant product is a map
$$\backslash\colon E^p(X) \otimes F_q(X \wedge Y) \to (E \wedge F)_{-p+q}(Y)$$
$$f \otimes g \mapsto \mathbb{S} \xrightarrow{g} F \wedge X \wedge Y \cong X \wedge F \wedge Y \xrightarrow{f \wedge 1} E \wedge F \wedge Y$$
for $X \wedge Y \xrightarrow{f} E \in E^p(X \wedge Y), \mathbb{S} \xrightarrow{g} F \wedge Y \in F_q(Y)$

Proposition 2.2. *The four products are biadditive ($\mathbb{Z}$-bilinear), so in fact do give well defined products out of the tensor product.*

Again, we can relativise this by restricting along $(X, A) \mapsto \Sigma^\infty \left({}^{X}\!/\!{}_{A} \right)$. We also observe that ${}^{X}\!/\!{}_{A} \wedge {}^{Y}\!/\!{}_{B} \cong {}^{X \times Y}\!/\!{}_{(A \times Y \cup X \times B)}$.

Notation 2.3. For CW-pairs $(X, A), (Y, B)$ we set

$$(X, A) \times (Y, B) := (X \times Y, A \times Y \cup X \times B)$$

Proposition 2.4. *The four products relativise to products*

- $E^p(X, A) \otimes F^q(Y, B) \xrightarrow{\bar{\wedge}} (E \wedge F)^{p+q}((X, A) \times (Y, B))$
- $E_p(X, A) \otimes F_q(Y, B) \xrightarrow{\underline{\wedge}} (E \wedge F)_{p+q}((X, A) \times (Y, B))$
- $E^p((X, A) \times (Y, B)) \otimes F_q(Y, B) \xrightarrow{/} (E \wedge F)^{p-q}(X, A)$
- $E^p(X, A) \otimes F_q((X, A) \times (Y, B)) \xrightarrow{\backslash} (E \wedge F)_{-p+q}(Y, B)$

Notation 2.5. The construction of (co)homology is also functorial in maps of spectra of nonzero degree. Adams follows the following sign convention. For a map $f\colon X \to Y$ of degree d, we will define the action $f^*\colon E^n(Y) \to E^{n-d}(X)$ to act as $f^*(u) = (-1)^{dn} u \circ f$. The functoriality in E, as well as both kinds of functoriality for homology takes place by some form of post-composition and is not assigned a sign change.

Remark 2.6. We will ourselves be unaffected by this sign convention, as we will mostly be concerned with functoriality along degree zero maps (for instance twists). Regardless we describe the compatibility of the products with maps of nonzero degree in the following two propositions.

Proposition 2.7. *Fix $f\colon X \to X'$, $g\colon Y \to Y'$ of any degree (denoted $|f|, |g|$), then we have the following relations*

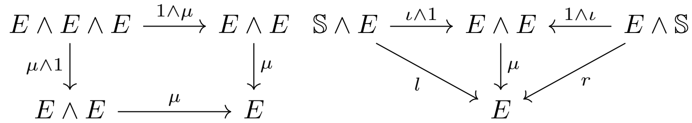
- *For $u \in E^*(X'), v \in F^*(Y')$ we have $(f \wedge g)^*(u \bar{\wedge} v) = (-1)^{|g||u|}(f^* u) \wedge (g^* v)$*
- *For $u \in E_*(X), v \in F_*(Y)$ we have $(f \wedge g)_*(u \underline{\wedge} v) = (-1)^{|g||u|}(f_* u) \wedge (g_* v)$*
- *For $u \in E^*(X' \wedge Y'), v \in F_*(Y)$ we have $((f \wedge g)^* u)/v = (-1)^{|g||u|} f^*((u/g)_* v)$*
- *For $u \in E^*(X'), v \in F_*(X \wedge Y)$ we have $u \backslash ((f \wedge g)_* v) = (-1)^{|g||f|+|g||u|+|f||u|} g_*((f^* u) \backslash v)$*

Proposition 2.8. *For $e\colon E \to E', f\colon F \to F'$ of any degree we have*

$$(e \wedge f)_*(u \star v) = (-1)^{|f||u|}(e_*(u) \star f_*(v))$$

where $\star$ is any one of the four product operations and u, v are such that the equation is well formed.

Definitions 2.9. A ring spectrum is a CW-spectrum E with the data of a "product" $\mu\colon E \wedge E \to E$ and "identity" $\iota\colon \mathbb{S} \to E$ such that the following diagrams commute in $\mathsf{hCW\text{-}Spec}$.



A ring spectrum E (or more precisely (E, μ, ι)) is further commutative if

$$\begin{array}{ccc} E \wedge E & \xrightarrow{\quad c \quad} & E \wedge E \\ & {\scriptstyle\mu}\searrow \quad \swarrow{\scriptstyle\mu} & \\ & E & \end{array}$$

A module over a ring spectrum E is a spectrum F with the data of an "action" $\alpha\colon E \wedge F \to F$ such that

$$\begin{array}{ccc} E \wedge E \wedge F & \xrightarrow{1\wedge\alpha} & E \wedge F \\ {\scriptstyle\mu\wedge 1}\downarrow & & \downarrow{\scriptstyle\alpha} \\ E \wedge F & \xrightarrow{\quad\alpha\quad} & F \end{array} \qquad \begin{array}{ccc} \mathbb{S} \wedge F & \xrightarrow{\iota\wedge 1} & E \wedge F \\ & {\scriptstyle l}\searrow & \downarrow{\scriptstyle\alpha} \\ & & F \end{array}$$

A ring spectrum E gives us through the pairing μ products

- $\bar{\wedge}\colon E^p(X) \otimes E^q(Y) \to E^{p+q}(X \wedge Y)$
- $\underline{\wedge}\colon E_p(X) \otimes E_q(Y) \to E_{p+q}(X \wedge Y)$
- $/\colon E^p(X \wedge Y) \otimes E_q(Y) \to E^{p-q}(X)$
- $\backslash\colon E^p(X) \otimes E_q(X \wedge Y) \to E_{-p+q}(Y)$

with analogous relative versions of the same.

Remark 2.10. Of course, for a module F over a ring spectrum E we get in the same way products

- $\underline{\wedge}\colon E^p(X) \otimes F^q(Y) \to F^{p+q}(X \wedge Y)$
- $\underline{\wedge}\colon E_p(X) \otimes F_q(Y) \to F_{p+q}(X \wedge Y)$
- $/\colon E^p(X \wedge Y) \otimes F_q(Y) \to F^{p-q}(X)$
- $\backslash\colon E^p(X) \otimes F_q(X \wedge Y) \to F_{-p+q}(Y)$

and analogous relative versions of the same.

Our goal will be to show that for a CW-complex X and a commutative ring spectrum E, $E^*(X) := E^*(X, \emptyset)$ is canonically a graded commutative ring such that for a subcomplex A, $E_*(X, A)$ and $E^*(X, A)$ are both graded modules over it. We now define precisely which operations give us the ring operations and the module action.

Definitions 2.11. For E, F spectra and CW-pairs $(X, A), (X, B)$ the diagonal map induces a map of pairs

$$\delta\colon (X, A \cup B) \to (X \times X, A \times X \cup X \times B) = (X, A) \times (Y, B)$$

We define the "cup product" as the composite

$$\smile\colon E^p(X,A) \otimes F^q(Y,B) \xrightarrow{\bar{\wedge}} (E \wedge F)^{p+q}(X \times X, A \times X \cup X \times B) \xrightarrow{\delta^*} (E \wedge F)^{p+q}(X, A \cup B)$$

and the "cap product" as the composite

$$\frown\colon E^p(X,A) \otimes F_q(X, A \cup B) \xrightarrow{1\otimes\delta_*} E^p(X,A) \otimes F_q(X \times X, X \times B \cup A \times X) \xrightarrow{\backslash} (E \wedge F)_{-p+q}(X, B)$$

In particular, when E is a ring spectrum we get maps

$$E^p(X, A) \otimes E^q(X, B) \xrightarrow{\smile} E^{p+q}(X, A \cup B)$$

$$E^p(X, A) \otimes E_q(X, A \cup B) \xrightarrow{\frown} E_{-p+q}(X, B)$$

Remarks 2.12. As a special case, setting $A = \phi$ we get maps $E^*(X) \otimes E^*(X) \xrightarrow{\smile} E^*(X), E^*(X) \otimes E^*(X, A) \xrightarrow{\smile} E^*(X, A), E^*(X) \otimes E_*(X, A)$. These maps will be our ring multiplications and monoid actions. Note also that this definition does not directly extend to spectra as there is no obvious "diagonal".

Remark 2.13. If we denote by $X \xleftarrow{\pi_X} X \times Y \xrightarrow{\pi_Y} Y$ the projections, then we have commuting diagrams

$$\begin{array}{ccc}
E^p(X,A) \otimes F^q(Y,B) & \xrightarrow{\bar{\wedge}} & (E \wedge F)^{p+q}(X \times Y, X \times B \cup A \times Y) \\
& \searrow^{\pi_X^* \otimes \pi_Y^*} & \uparrow \smile \\
& & E^p(X \times Y, A \times Y) \otimes F^q(X \times Y, X \times B)
\end{array}$$

$$\begin{array}{ccc}
E^p(X,A) \otimes F_q(X \times Y, A \times Y \cup X \times B) & \xrightarrow{\quad \backslash \quad} & (E \wedge F)_{-p+q}(Y,B) \\
\downarrow \pi_X^* \otimes 1 & & (\pi_Y)_* \uparrow \\
E^p(X \times Y, A \times Y) \otimes F_q(X \times Y, A \times Y \cup X \times B) & \xrightarrow{\frown} & (E \wedge F)_{-p+q}(X \times Y, X \times B)
\end{array}$$

so in particular (at least in the relative case) the products $\bar{\wedge}, \backslash$ can be recovered from the cup and cap products.

We will deduce the validity of the ring multiplication and module action axioms from general properties of the four "multiplications" we have defined. They satisfy a wide range of associativity relations, reminiscent of various rules for "cancelling fractions".

Proposition 2.14. *For spectra E, F, G, X, Y, Z, we have:*

(1) For $u \in E^p(X), v \in F^q(Y), w \in G^p(Z)$

$$(u \bar{\wedge} v) \bar{\wedge} w = u \bar{\wedge} (v \bar{\wedge} w) \in (E \wedge F \wedge G)^{p+q+r}(X \wedge Y \wedge Z)$$

(2) For $y \in E^p(Y), x \in F^q(X), t \in G_r(X \wedge Y \wedge Z)$

$$y \backslash (x \backslash t) = (c^*(y \underline{\wedge} x)) \backslash t \in (E \wedge F \wedge G)_{-p-q+r}(Z)$$

(3) For $u \in E_p(X), v \in F_q(Y), w \in G_p(Z)$

$$(u \underline{\wedge} v) \underline{\wedge} w = u \underline{\wedge} (v \underline{\wedge} w) \in (E \wedge F \wedge G)_{p+q+r}(X \wedge Y \wedge Z)$$

(4) For $x \in E^p(X), u \in F^q(Y \wedge Z), z \in G_r(Z)$

$$x \bar{\wedge} (u/z) = (x \bar{\wedge} u)/z \in (E \wedge F \wedge G)^{p+q-r}(X \wedge Y)$$

(5) For $v \in E^p(X \wedge Z), y \in F^q(Y), u \in G_r(Y \wedge Z)$

$$v/(y \backslash u) = ((1 \wedge c)^*(v \bar{\wedge} y))/y \in (E \wedge F \wedge G)^{p+q-r}(X)$$

(6) For $t \in E^p(X \wedge Y \wedge Z), z \in F^q(Z), y \in G_r(Y)$

$$(t/z)/y = t/(c_*(z \underline{\wedge} y)) \in (E \wedge F \wedge G)^{p-q-r}$$

(7) For $w \in E^p(X \wedge Y), y \in F_q(Y), v \in G_r(X \wedge Z)$

$$(w/y) \backslash v = w \backslash ((c \wedge 1)_*(y \underline{\wedge} v)) \in (E \wedge F \wedge G)_{-p+q+r}(Z)$$

(8) For $x \in E^p(X), w \in F_q(X \wedge Y), z \in G_z(Z)$

$$(x \backslash w) \underline{\wedge} Z = x \backslash (w \underline{\wedge} Z) \in (E \wedge F \wedge G)_{-p+q+r}(Y \wedge Z)$$

Remark 2.15. When we extend this using pairings of spectra, we get similar equalities as long as we use pairings such that

$$\begin{array}{ccc} E \wedge F \wedge G & \longrightarrow & H \wedge G \\ \downarrow & & \downarrow \\ E \wedge K & \longrightarrow & L \end{array}$$

(so that the statement even makes sense) such as with a (commutative) ring spectrum.

We will primarily be interested in the first two properties. The first shows that the cup product is an associative multiplication on $E^*(X)$ as well as an associative action on $E^*(X, A)$. The second will show that the cap product is an associative action of $E^*(X)$ on $E_*(X, A)$.

Proposition 2.16. *We have the following relations on relative groups obtained from the associativity relations for the external and slant products.*

(1) For $x \in E^*(X, A), y \in F^*(X, B), z \in G^*(X, C)$ *we have*

$$(x \smile y) \smile z = x \smile (y \smile z) \in (E \wedge F \wedge G)^*(X, A \cup B \cup C)$$

(2) For $x \in E^*(X, A), y \in F^*(X, B), z \in G^*(X, A \cup B \cup C)$ *we have*

$$(x \smile y) \frown z = x \frown (y \frown z) \in (E \wedge F \wedge G)_*(X, C)$$

Thus in particular the multiplication on $E^*(X)$ *and the* $E^*(X)$*-actions on* $E^*(X, A)$ *and* $E_*(X, A)$ *satisfy the corresponding associativity/distributivity properties.*

Proof. The proofs are mostly symbolic consequences of the associativity relations. For instance for (2) we have

$$x \frown (y \frown z) := x\backslash\delta_*(y \frown z) = x\backslash(\delta_*(y\backslash\delta_* z))$$

by unraveling definitions and by our functoriality observations we may rewrite this as

$$x\backslash(\delta_*(y\backslash\delta_* z)) = x\backslash(y\backslash((1 \times \delta)_*\delta_* z)) = c^*(x\bar{\wedge}y)\backslash(\delta \times 1)_*\delta_* z = \delta^* c^*(x\bar{\wedge}y)\backslash\delta_* z$$

with the middle equality being part (2) of the result for exterior products. But $c\delta = \delta$ so this is simply

$$\delta^*(x\bar{\wedge}y)\backslash\delta_* z =: (x \smile y) \frown z$$

□

Definition 2.17. For a ring spectrum (E, μ, ι), set for a space X

$$1\colon \Sigma^\infty X_+ \cong (\Sigma^\infty X_+) \wedge \mathbb{S} \xrightarrow{\Sigma^\infty p_+ \wedge \iota} \mathbb{S} \wedge E \cong E$$

where p is the unique $X_+ \to *$. This defines $1 \in E^0(X)$.

Lemma 2.18. *For* $x \in E^n(X, A), y \in E^m(Y, B)$ *we have* $\pi_X^*(x) = x\bar{\wedge}1$ *and* $\pi_Y^*(y) = 1\bar{\wedge}y$ *where* $X \leftarrow X \times Y \to Y$ *are the projections. (note that the* 1 *in* $x\bar{\wedge}1$ *is* $1 \in E^0(Y)$ *and the one in* $1\bar{\wedge}y$ *is* $1 \in E^0(X)$*)*

Corollary 2.19. $1 \smile x = x = x \smile 1$ *and* $1 \frown y = y$ *for* $x \in E^*(X, A), y \in E_*(Y, B)$*. That is, both satisfy the required unitality properties.*

It remains only to address graded commutativity of $E^*(X)$. Ultimately this will follow from the compatibility of $\bar{\wedge}$ with the twist maps.

Proposition 2.20. *We have for spectra E, F, X, Y the following commutative diagrams (up to the displayed sign $(-1)^{pq}$)*

$$\begin{array}{ccccc}
E^p(X)\otimes F^q(Y) & & \xrightarrow{\quad\simeq\quad} & & F^q(Y)\otimes E^p(X) \\
\downarrow{\scriptstyle \bar\wedge} & & & & \downarrow{\scriptstyle \bar\wedge} \\
(E\wedge F)^{p+q}(X\wedge Y) & & (-1)^{pq} & & (F\wedge E)^{p+q}(Y\wedge X) \\
& \searrow{\scriptstyle c^*_{YX}} & & \nearrow{\scriptstyle (c_{EF})_*} & \\
& & (E\wedge F)^{p+q}(Y\wedge X) & &
\end{array}$$

$$\begin{array}{ccc}
E_p(X)\otimes F_q(Y) & \xrightarrow{\quad\simeq\quad} & F_q(Y)\otimes E_p(X) \\
\downarrow{\scriptstyle \underline{\wedge}} & (-1)^{pq} & \downarrow{\scriptstyle \underline{\wedge}} \\
(E\wedge F)_{p+q}(X\wedge Y) & \xrightarrow{(c_{EF}\wedge c_{XY})_*} & (F\wedge E)_{p+q}(Y\wedge X)
\end{array}$$

Consequently, extending the two exterior products using pairings $E\wedge F\to G$ and $F\wedge E\to G$ such that

$$\begin{array}{ccc}
E\wedge F & \xrightarrow{\quad c\quad} & F\wedge E \\
& \searrow \quad \swarrow & \\
& G &
\end{array}$$

(for instance the multiplication $\mu\colon E\wedge E\to E$ of a commutative ring spectrum) we have for $x\in E^p(X), y\in F^q(Y)$, $x\bar\wedge y = (-1)^{pq} y\bar\wedge x\in G^{p+q}(X\wedge Y)$. Thus in particular $E^*(X)$ is indeed graded commutative, as desired.

Proof. [Switzer, Prop 13.53] sketches a proof for the case of homology, so we sketch a proof for the case of cohomology (which is the case we use). For this we chase a simple tensor $x\otimes y$, for $x\in E^p(X)\cong[X,\mathbb{S}^p\wedge E], y\in F^q(Y)\cong[Y,\mathbb{S}^q\wedge F]$. The result follows from the commutativity of the diagram

$$\begin{array}{ccccccc}
Y\wedge X & \xrightarrow{y\wedge x} & \mathbb{S}^q\wedge F\wedge\mathbb{S}^p\wedge E & \xrightarrow{1\wedge c\wedge 1} & \mathbb{S}^q\wedge\mathbb{S}^p\wedge F\wedge E & \xrightarrow{\simeq} & \mathbb{S}^{p+q}\wedge F\wedge E \\
{\scriptstyle c}\uparrow & & {\scriptstyle c_{\mathbb{S}^p\wedge E,\mathbb{S}^q\wedge F}}\uparrow & & {\scriptstyle c\wedge c}\uparrow & & {\scriptstyle (-1)^{pq}\wedge c}\uparrow \\
X\wedge Y & \xrightarrow{x\wedge y} & \mathbb{S}^p\wedge E\wedge\mathbb{S}^q\wedge F & \xrightarrow{1\wedge c\wedge 1} & \mathbb{S}^p\wedge\mathbb{S}^q\wedge E\wedge F & \xrightarrow{\simeq} & \mathbb{S}^{p+q}\wedge E\wedge F
\end{array}$$

where the first two squares commute purely by the symmetric monoidality of $\wedge$ and the final square due to the fact that

$$\begin{array}{ccc}
\mathbb{S}^q\wedge\mathbb{S}^p & \xrightarrow{\simeq} & \mathbb{S}^{p+q} \\
{\scriptstyle c}\uparrow & & \uparrow{\scriptstyle (-1)^{pq}} \\
\mathbb{S}^p\wedge\mathbb{S}^q & \xrightarrow{\simeq} & \mathbb{S}^{p+q}
\end{array}$$

□

References

[Adams] J. F. Adams, *Stable homotopy and generalised homology*, Part III, The University of Chicago Press, 1974

[Switzer] R. M. Switzer, *Algebraic Topology - Homotopy and Homology*, Springer, 1975