RESOLUTIONS, HOMOLOGY AND COHOMOLOGY SEMINAR: *HOMOTOPICAL ALGEBRA - MODEL CATEGORIES*

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1. Homology and Cohomology in a Model Category

Classically, for an abelian category $\mathcal{A}$ one defines the homology functors $\mathsf{Ch}_{\geq 0}(\mathcal{A}) \xrightarrow{H_n} \mathcal{A}$ by a "Kernel/Image" construction.

Proposition 1.1. *(Dold-Kan) For a commutative ring R, there is an equivalence of categories $s\mathsf{Mod}_R \simeq \mathsf{Ch}_{\geq 0}(R)$.*

The idea is to use the category of simplicial objects $s\mathcal{C}$ as a replacement for $\mathsf{Ch}_{\geq 0}(\mathcal{C})$, as it makes sense for any category.

Definition 1.2. The data of an abelian object in a category $\mathcal{C}$ is that of an object a and the structure of a functorial abelian group structure on $よ_{\mathcal{C}}(a)$, that is a lift of $\mathcal{C}(-, a) \colon \mathcal{C}^{\mathsf{op}} \to \mathsf{Set}$ to Ab. When $\mathcal{C}$ has finite limits, this is equivalent by the よ-lemma to morphisms $* \xrightarrow{0} a, a \times a \xrightarrow{m} a$ and $a \xrightarrow{i} a$ making the diagrams encoding associativity, unitality and inverses commute.

We will denote by $\mathcal{C}_{ab}$ the subcategory of $\mathsf{Fun}(\mathcal{C}^{\mathsf{op}}, \mathsf{Ab})$ spanned by the abelian group objects $よ_{\mathcal{C}}(a)$.

Assumption 1.3. We will assume that the faithful map $\mathcal{C}_{ab} \xrightarrow{U} \mathcal{C}$ has a left adjoint, called "abelianisation" (denoted Ab) and further that $\mathcal{C}$ and $\mathcal{C}_{ab}$ are given model structures making this into a Quillen adjunction.

Remark 1.4. As model categories have (finite) limits, one can check (using the fact that products in the category are products in the homotopy category and so on) that an abelian object in a model category is naturally an abelian object in its homotopy category.

Definition 1.5. We define the Quillen homology to be the left derived functor $\mathsf{L}Ab$ (left adjoint to the right derived functor $\mathsf{R}U$) of the abelianisation functor. This yields a functor $\mathsf{ho}\,\mathcal{C} \to \mathsf{ho}\,\mathcal{C}_{ab}$. We may further define the n-th Quillen homology groups by taking π_n of the Quillen homology.

Observation 1.6. *$\mathcal{C}_{ab}$ is pointed (the unique morphism from the terminal $よ_{\mathcal{C}}(*)$ to any other abelian object is the identity of the group structure), so there are suspension and loop operators Σ, Ω on $\mathsf{ho}\,\mathcal{C}_{ab}$.*

Assumption 1.7. Assume that the unit $1 \xrightarrow{\eta} \Omega\Sigma$ is an isomorphism.

Definition 1.8. For an object A of $\mathcal{C}_{ab}$ we define the p-th Quillen cohomology functor with coefficients in A by $H^p(X; A) = [\mathsf{L}Ab(-), \Omega^{p+n}\Sigma^n]$, for $n \geq 0$ chosen such that $p + n \geq 0$. This is well defined by the preceding assumption.

Assumption 1.9. Assume that $\mathcal{C}$ is pointed. Thus $\mathsf{ho}\,\mathcal{C}$ has its own suspension and loop operators as well.

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Proposition 1.10. *The left derived functor* $\mathsf{L}Ab$ *commutes with suspension (and dually, the right derived functor* $\mathsf{R}U$ *commutes with looping).*

Corollary 1.11. *We have natural suspension isomorphisms* $H^n(\Sigma X; A) \cong H^{n+1}(X; A)$.

Remark 1.12. Recall that in a pointed model category, the representable functors on the homotopy category produce from a cofibre sequence a long exact sequence. Consequently, we have a long exact sequence in cohomology associated to a cofibre sequence.

Motivated by this, we call an object in the essential image of $\mathsf{R}U$ a generalised Eilenberg-Maclane object, as it and its loopings represent a series of functors that resembles a cohomology theory in the sense of Eilenberg-Steenrod.

2. Examples of Quillen Homology

2.1. **Singular Homology.** $\mathsf{Set}_{ab} = \mathsf{Ab}$ and the abelianisation is given by the free abelian group functor $\mathbb{Z}^{\oplus(-)} \colon \mathsf{Set} \to \mathsf{Ab}$. Passing to simplicial objects, we have the abelianisation $\mathbb{Z}^{\oplus(-)} \colon s\mathsf{Set} \to s\mathsf{Ab} = (s\mathsf{Set})_{ab}$

As every simplicial set is cofibrant, we can compute it as $\mathsf{L}Ab(X) = \mathbb{Z}^{\oplus X}$. On the singular complex of a space $\mathsf{Sing}(X)$, this simply computes the singular homology as $\pi_n \mathsf{L}Ab\,\mathsf{Sing}(X) = \pi_n \mathbb{Z}^{\oplus\,\mathsf{Sing}(X)} \cong H_n(X;\mathbb{Z})$. (The last identification is because under the Dold-Kan correspondence, homotopy corresponds to homology)

2.2. **Group Homology.** We have $\mathsf{Grp}_{ab} = \mathsf{Ab}$ as well, with the abelianisation being given by the assignment $X \mapsto X/[X,X]$. Thus we have an abelianisation $\mathsf{L}Ab \colon s\mathsf{Grp} \to s\mathsf{Ab}$ computed as $\mathsf{L}Ab(G) = X/[X,X]$ (levelwise quotient) where $X \xrightarrow{\sim} G$ is a cofibrant replacement.

Recall that the group homology of G is defined as $H_n(\mathsf{B}G) = \mathsf{Tor}_n^{\mathbb{Z}[G]}(\mathbb{Z},\mathbb{Z})$. One can compute this in terms of the Quillen Homology of G thought of as a constant object in $s\mathsf{Grp}$. To be precise, one obtains through a spectral sequence argument upon considering the two canonical filtrations of $\mathbb{Z}^{\oplus BG}$ that there is a canonical $\pi_n \mathsf{L}Ab(G) \cong H_{n+1}(BG)$ for each n.

2.3. **André-Quillen Homology.** For a commutative ring R and a commutative R-algebra A, the abelian objects in CAlg_R/A can be identified with A-modules. Moreover, the hom objects in the adjunction bijection can be naturally identified with the values of the R-derivations functor. The associated Quillen homology, called André-Quillen Homology can be used to study the derived functors of R-derivations and is one of the first major applications of model categorical methods.

Definition 2.1. For an $A \in \mathsf{CAlg}_R$ and $M \in \mathsf{Mod}_A$, we let $A \ltimes M$ be the A-module $A \oplus M$ with a multiplication being given by $(a,m)\cdot(a',m') = (aa', am' + a'm)$. Inclusion onto the A component makes it an A-algebra (and hence R-algebra) and projection onto the first makes it an object of CAlg_R/A.

Proposition 2.2. *Projection to the second component defines an isomorphism (functorially in* X, M*)* $\mathsf{CAlg}_R/A(X, A \ltimes M) \cong \mathsf{Der}_R(X, M)$. *In particular,* $A \ltimes M$ *is an abelian object.*

Proposition 2.3. *The functor* $A \ltimes (-)$ *defines an equivalence* $\mathsf{Mod}_A \xrightarrow{\sim} (\mathsf{CAlg}_R/A)_{ab}$. *Hence we also have an equivalence* $s\mathsf{Mod}_A \simeq (s\mathsf{CAlg}_R/A)$.

Definition 2.4. For $A \in \mathsf{CAlg}_R$, let I be the kernel of the multiplication $A \otimes_R A \to A$. Then we define $\Omega_{A/R} := I/I^2$ to be the module of (Kähler) differentials. There is a canonical map $d \colon A \to \Omega_{A/R}$ acting as $a \mapsto a \otimes 1 - 1 \otimes a$.

Proposition 2.5. *Pulling back along d induces a natural* $\mathsf{Mod}_A(\Omega_{A/R}, -) \cong \mathsf{Der}_R(A, -)$. *Thus, we have a chain of natural isomorphisms*

$$\mathsf{Mod}_A(A \otimes_X \Omega_{X/R}, M) \cong \mathsf{Mod}_X(\Omega_{X/R}, M) \cong \mathsf{Der}_R(X, M) \cong \mathsf{CAlg}_R/A(X, A \ltimes M)$$

Corollary 2.6. *The adjunction $A \otimes_{(-)} \Omega_{(-)/R} \dashv A \ltimes (-)$ is an abelianisation adjunction. (It is Quillen as the right adjoint $A \ltimes (-)$ preserves weak equivalences and fibrations)*

Definition 2.7. For $A \in \mathsf{CAlg}_R$, we call $L_{A/R} := \mathsf{L}Ab(A)$ the cotangent complex of A and define the André-Quillen Homology functor with coefficients in an A-module M to be $D_n \left({}^{A}\!/_{R}; M \right)(-) := \pi_n(\mathsf{L}Ab(-) \otimes^L_A M)$.

Observation 2.8. *One observes that the associated Quillen cohomology groups in the sense of the first section can be described as* $D^n \left({}^{A}\!/_{R}; M \right)(-) \cong [-, A \ltimes K(M, n)]$, *where $K(M, n)$ can also be described as the simplicial A-module corresponding to the chain complex of A-modules which is M in degree n and zero elsewhere.*

3. Resolution Model Categories

The category of simplicial objects in a model category can always be endowed with the Reedy model structure, but in practice one may desire a perturbation of this structure. For instance, cofibrant replacements often play the role of "resolutions", and one would like these resolutions to be built out of certain predetermined "projectives".

Furthermore, one constructs certain weak equivalence invariants (for instance, many spectral sequences) and we would like these invariants to be more complete, for instance to detect weak equivalences. These can be thought of as the Reedy structure having too many cofibrant objects and not enough weak equivalences.

Assumption 3.1. We will work in a pointed simplicial model category $s\mathcal{C}$.

Remark 3.2. We will identify an object of $\mathcal{C}$ with the corresponding constant object of $s\mathcal{C}$.

Definitions 3.3. The data of a cogroup object in a category $\mathcal{C}$ is that of an object a and the structure of a functorial group structure on $よ^{\mathcal{C}}(a)$, that is a lift of $\mathcal{C}(a, -) \colon \mathcal{C} \to \mathsf{Set}$ to Grp. Again, when $\mathcal{C}$ has finite coproducts this structure can be internalised to that of a collection of morphisms of $\mathcal{C}$ such that the appropriate diagrams commute.

A homotopy cogroup object in a model category is a **cofibrant** object and the structure of a cogroup object on that object in the homotopy category.

When the model category is pointed, we will call a collection of homotopy cogroup objects closed under finite coproducts and suspensions a class of projectives.

Example 3.4. In the pointed simplicial model category $s\mathsf{Mod}_R$, every cofibrant object is canonically a homotopy cogroup object. Important in practice is the canonical set of projectives given by finite sums of objects of the form $\Sigma^n R$.

Definitions 3.5. Consider a pointed simplicial model category $\mathcal{C}$ and $\mathcal{P}$ a set of projectives in it. Then:

- Call a morphism $X \to Y$ in $\mathsf{ho}\mathcal{C}$ a $\mathcal{P}$-epi if for every $P \in \mathcal{P}$, the induced map $[P, X] \to [P, Y]$ is a surjection.
- Call an object A of $\mathcal{C}$ $\mathcal{P}$-proj if for each $\mathcal{P}$-epi $X \to Y$, the induced map $[A, X] \to [A, Y]$ is a surjection.
- Call a morphism $A \to B$ in $\mathcal{C}$ a $\mathcal{P}$-proj cofibration if it has the left lifting property with fibrations of $\mathcal{C}$ that are also $\mathcal{P}$-epi.

Remarks 3.6. We have immediately that (note that we assumed homotopy cogroup objects to be cofibrant)

- The classes $\mathcal{P}$-epi and $\mathcal{P}$-proj determine each other.
- Every element of $\mathcal{P}$ is $\mathcal{P}$-proj.
- $\mathcal{P}$-proj is closed under arbitrary coproducts.

Lemma 3.7. *Given a class of projectives,* $\mathsf{ho}\,\mathcal{C}$ *has enough projectives, that is every object* X *is the target of a* $\mathcal{P}$*-epi* $Y \to X$ *with* Y $\mathcal{P}$*-proj.*

One can construct such a map by considering $Y := \coprod_{P \in \mathcal{P}} \coprod_{P \to X} P$ *and defining the map by taking the restriction to a component to be its index.*

Definition 3.8. Call a morphism $X \to Y$ in $\mathcal{C}$ $\mathcal{P}$-free if it factors as $X \to X \coprod F \to Y$ where F is cofibrant and $\mathcal{P}$-projective, and the second map is a trivial cofibration. (the first map is the component inclusion)

Proposition 3.9. *A morphism in* $\mathcal{C}$ *is a* $\mathcal{P}$*-projective cofibration if and only if it is a retract of a* $\mathcal{P}$*-free map.*

Definitions 3.10. Consider a pointed simplicial model category $\mathcal{C}$, a class of projectives $\mathcal{P}$ and a morphism $f\colon X \to Y$ in $s\mathcal{C}$. Then we say that:

- f is a $\mathcal{P}$-eq if for all $P \in \mathcal{P}$, the induced $[P, X] \to [P, Y]$ is a weak equivalence of simplicial groups, or equivalently a levelwise weak equivalence.
- f is a $\mathcal{P}$-fibration if it is a Reedy fibration, and further the induced map $[P, X] \to [P, Y]$ is a fibration of simplicial groups.
- f is a $\mathcal{P}$-cofibration if the relative latching maps $X_n \coprod_{L_n X} L_n Y \to Y_n$ are all $\mathcal{P}$-projective cofibrations.

Remarks 3.11. Every Reedy weak equivalence is a levelwise weak equivalence, so they introduce isomorphisms on the hom objects of the homotopy category, and are thus all $\mathcal{P}$-eqs.

Further, every $\mathcal{P}$-cofibration is necessarily a Reedy cofibration. Also, an object is $\mathcal{P}$-fibrant if and only if it is Reedy fibrant.

Theorem 3.12. *For a pointed simplicial model category* $\mathcal{C}$ *and a class of projectives* $\mathcal{P}$*, the* $\mathcal{P}$*-eqs,* $\mathcal{P}$*-fibrations and the* $\mathcal{P}$*-cofibrations are the weak equivalences, fibrations and cofibrations respectively of a (simplicial) model structure on* $s\mathcal{C}$*, called the* $\mathcal{P}$*-resolution model structure.*

Furthermore, this is cofibrantly generated if $\mathcal{C}$ *is.*

Example 3.13. In the category CGH_* of pointed compactly generated Hausdorff spaces, every suspension is a homotopy cogroup object via the pinch map. Of particular interest is that class of projectives generated by an object X, that is the class of finite wedges of suspensions $\Sigma^n X$. The prototypical example is when X is S^0, that is the class of finite wedges of spheres.

In this case, a cofibrant replacement $Y \to X$ can be thought of as a "resolution by spheres". In particular, passing to the realisation, the induced $|Y| \to X$ defines a weak equivalence of $|Y|$ onto the connected component of X containing the basepoint. (so if X is connected, we get $|Y| \xrightarrow{\sim} X$)

If for instance we used instead $X := S^k$, to only consider spheres of dimension at least that k, the so constructed $|Y|$ would model instead the $(k-1)$-connected cover or k-coskeleton of X.